Loop Integrals & Thm. on Resolution of Singularities
（ループ積分と特異点解消定理）

2011. Jan. 7

Reference List

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- Zeillinger L'Enseignement Mathématique 52 (2006) 143
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1. Sector decomposition of Feynman param. integrals

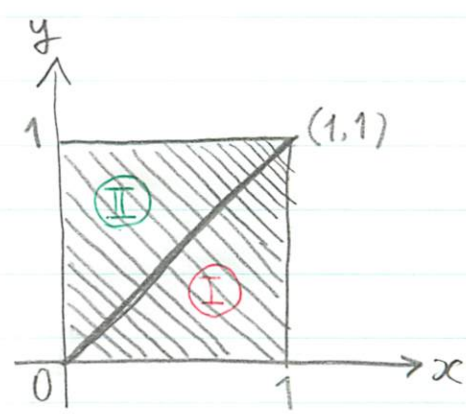
$\Rightarrow$ Factorization of entangled singularities

Factorized singularity is easily computable (numerically):

$$\int_0^1 dx\ x^{\epsilon-1} \underbrace{f(x)}_{\text{reg. as } x\to 0} \qquad \text{log div. as } \epsilon \to 0.$$

$$= \int_0^1 dx\ x^{\epsilon-1}\left[f(x)-f(0)\right] + f(0)\underbrace{\int_0^1 dx\ x^{\epsilon-1}}_{=\left[\frac{x^\epsilon}{\epsilon}\right]_0^1 = \frac{1}{\epsilon}}$$

$$= \frac{f(0)}{\epsilon} + \int_0^1 dx\ \underbrace{x^\epsilon\left[\frac{f(x)-f(0)}{x}\right]}_{\text{expand in } \epsilon \text{ before integ.}}$$



Entangled singularities $\Rightarrow$ sector decomp.

$$\underbrace{\int_0^1 dx \int_0^1 dy}\ x^\epsilon y^\epsilon \frac{\overset{\leftarrow\text{reg.}}{f(x,y)}}{xy(x+y)^\epsilon}$$

$$\int_0^1 dx \int_0^x dy + \int_0^1 dy \int_0^y dx \quad \text{II}$$

I

$x > y$ $x < y$

$y = xz$ $x = yw$

$\Downarrow$

$x+y = x(1+z)$

$$\int_0^1 dx \int_0^1 dz\ x^\epsilon z^\epsilon \frac{\boxed{f(x, xz)}}{xz\boxed{(1+z)^\epsilon}} \qquad \text{reg. as } x, z \to 0.$$

Singularities factorize.

* Repeat sector decomposition until all singularities factorize.

* Easy to automatize.

Applications (so far)

- $e^+e^- \to 2j$ NNLO differential distr.
- $gg \to H+X$ " "
  $\hookrightarrow (\gamma\gamma, WW)$
- $pp, p\bar{p} \to W, Z + X$ NNLO differential distr.
  $\hookrightarrow l_1 + \bar{l}_2$
- quark-gluon form factors in splitting fns. 3-loop
- QCD pot. 3-loop

⋮

- many publically available programs

Problem in complicated computations:

How to choose sub-sectors?

Problematic example:

$$P(x,y,z) = xz^2 + y^2 + yz$$

Maybe choice (x,y) is good, since $P(0,0,z)=0$

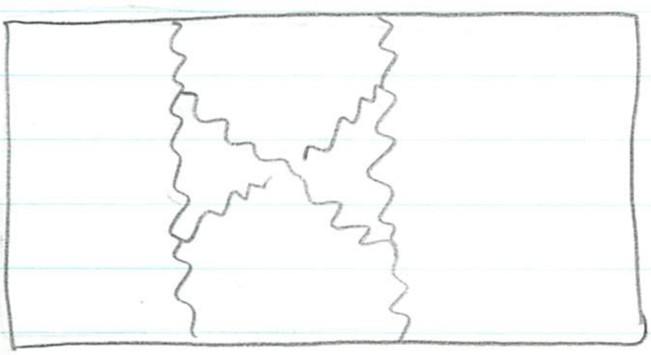
In $x>y$, $y=xw$

$$P = xz^2 + (xw)^2 + (xw)z$$

$$= x(\underbrace{z^2 + xw^2 + wz}_{= P(x,z,w)})$$

[If you choose (y,z), the algorithm terminates eventually.]

My problem:



Out[216]=

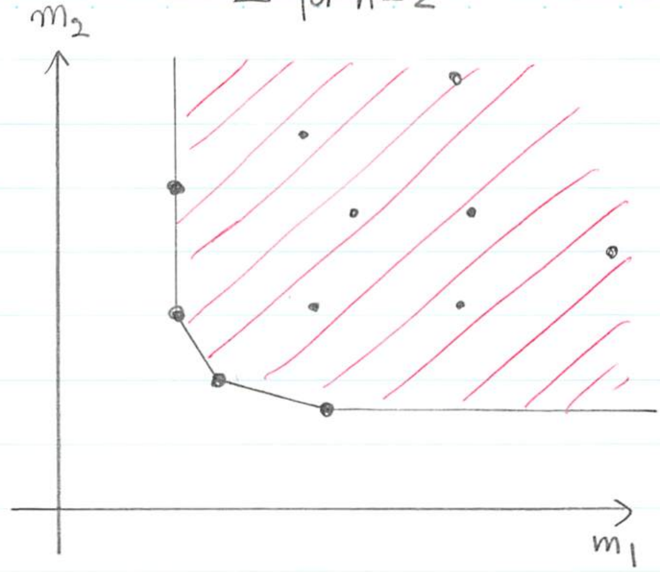
$$
\begin{aligned}
P_1 = \; & x_2 x_3 - x_2^2 x_3 - x_2 x_3^2 + x_2^2 x_3^2 + x_2^2 x_4 - x_2^3 x_4 - x_2 x_3 x_4 + 2 x_2^3 x_3 x_4 + \\
& x_2 x_3^2 x_4 - x_2^2 x_3^2 x_4 - x_2^3 x_3^2 x_4 - x_2^2 x_4^2 + x_2^3 x_4^2 + x_2^2 x_3 x_4^2 - \\
& 2 x_2^3 x_3 x_4^2 + x_2^3 x_3^2 x_4^2 - x_1 x_2 x_3 x_5 + x_1 x_2^2 x_3 x_5 + x_1 x_3^2 x_5 + \\
& x_1 x_2 x_3^2 x_5 - x_1 x_2^2 x_3^2 x_5 + 2 x_1 x_2 x_3 x_4 x_5 - 2 x_1 x_2^2 x_3 x_4 x_5 - \\
& 2 x_1 x_2 x_3^2 x_4 x_5 + 2 x_1 x_2^2 x_3^2 x_4 x_5 - x_1^2 x_3^2 x_5^2 - x_2 x_3 x_6 + x_1 x_2 x_3 x_6 + \\
& x_2^2 x_3 x_6 - x_1 x_2^2 x_3 x_6 + x_3^2 x_6 - x_1 x_3^2 x_6 + x_2 x_3^2 x_6 - x_1 x_2 x_3^2 x_6 - \\
& x_2^2 x_3^2 x_6 + x_1 x_2^2 x_3^2 x_6 + 2 x_2 x_3 x_4 x_6 - 2 x_1 x_2 x_3 x_4 x_6 - \\
& 2 x_2^2 x_3 x_4 x_6 + 2 x_1 x_2^2 x_3 x_4 x_6 - 2 x_2 x_3^2 x_4 x_6 + 2 x_1 x_2 x_3^2 x_4 x_6 + \\
& 2 x_2^2 x_3^2 x_4 x_6 - 2 x_1 x_2^2 x_3^2 x_4 x_6 - 2 x_1 x_3^2 x_5 x_6 + 2 x_1^2 x_3^2 x_5 x_6 - \\
& x_3^2 x_6^2 + 2 x_1 x_3^2 x_6^2 - x_1^2 x_3^2 x_6^2 + x_2 x_7 - 2 x_2^2 x_7 + x_2^3 x_7 + x_3 x_7 - \\
& 5 x_2 x_3 x_7 + 6 x_2^2 x_3 x_7 - 2 x_2^3 x_3 x_7 - x_3^2 x_7 + 4 x_2 x_3^2 x_7 - \\
& 4 x_2^2 x_3^2 x_7 + x_2^3 x_3^2 x_7 + 2 x_2 x_3 x_4 x_7 - 2 x_2^2 x_3 x_4 x_7 - 2 x_2 x_3^2 x_4 x_7 + \\
& 2 x_2^2 x_3^2 x_4 x_7 + 2 x_1 x_2 x_3 x_5 x_7 - 2 x_1 x_2^2 x_3 x_5 x_7 - 2 x_1 x_2 x_3^2 x_5 x_7 + \\
& 2 x_1 x_2^2 x_3^2 x_5 x_7 - 4 x_1 x_2 x_3 x_4 x_5 x_7 + 4 x_1 x_2^2 x_3 x_4 x_5 x_7 + \\
& 4 x_1 x_2 x_3^2 x_4 x_5 x_7 - 4 x_1 x_2^2 x_3^2 x_4 x_5 x_7 + 2 x_2 x_3 x_6 x_7 - \\
& 2 x_1 x_2 x_3 x_6 x_7 - 2 x_2^2 x_3 x_6 x_7 + 2 x_1 x_2^2 x_3 x_6 x_7 - 2 x_2 x_3^2 x_6 x_7 + \\
& 2 x_1 x_2 x_3^2 x_6 x_7 + 2 x_2^2 x_3^2 x_6 x_7 - 2 x_1 x_2^2 x_3^2 x_6 x_7 - 4 x_2 x_3 x_4 x_6 x_7 + \\
& 4 x_1 x_2 x_3 x_4 x_6 x_7 + 4 x_2^2 x_3 x_4 x_6 x_7 - 4 x_1 x_2^2 x_3 x_4 x_6 x_7 + \\
& 4 x_2 x_3^2 x_4 x_6 x_7 - 4 x_1 x_2 x_3^2 x_4 x_6 x_7 - 4 x_2^2 x_3^2 x_4 x_6 x_7 + \\
& 4 x_1 x_2^2 x_3^2 x_4 x_6 x_7 - x_2 x_7^2 + 2 x_2^2 x_7^2 - x_2^3 x_7^2 - x_3 x_7^2 + 4 x_2 x_3 x_7^2 - \\
& 5 x_2^2 x_3 x_7^2 + 2 x_2^3 x_3 x_7^2 + x_3^2 x_7^2 - 3 x_2 x_3^2 x_7^2 + 3 x_2^2 x_3^2 x_7^2 - x_2^3 x_3^2 x_7^2
\end{aligned}
$$

depends on 7 variables $(x_1, \cdots, x_7)$

2. Hironaka's polyhedra game
多面体

2 Players A & B are given a finite set M of points $\vec{m} = (m_1, \cdots, m_n) \in \mathbb{Z}^n_{\geq 0}$

Δ for $n=2$



Δ : Positive convex hull of M
凸包
= Newton polyhedron generated by M

$$\Delta \supseteq \bigcup_{m \in M} (\vec{m} + \mathbb{R}^n_{\geq 0})$$

A & B compete in the game:

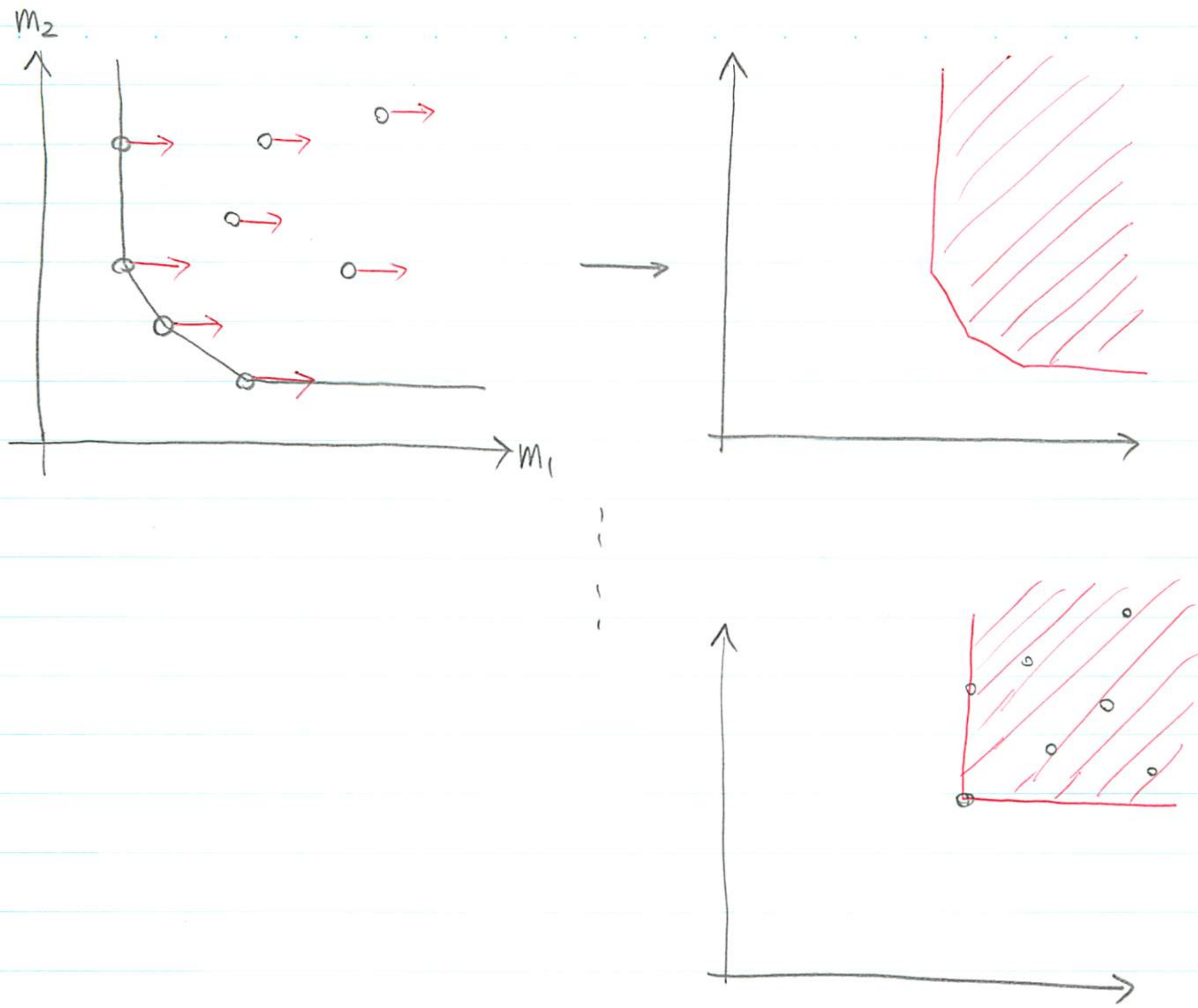
Repeat

1. Player A chooses $S = \{i, j\} \in \{1, \cdots, n\}$
2. Player B chooses either i or j from S, and replaces all $\vec{m} = (m_1, \cdots, m_n)$ of M by $\vec{m}' = (m_1', \cdots, m_n')$ given by

$$\begin{cases} m_k' = m_k & (k \neq i) \\ m_i' = m_i + m_j \end{cases} \qquad \begin{cases} m_k' = m_k & (k \neq j) \\ m_j' = m_i + m_j \end{cases}$$

Player A wins the game, if after a finite number of moves, Δ is of the form

$$\Delta = \vec{m} + \mathbb{R}^n_{\geq 0}$$



It can be shown that player A always has a winning strategy, no matter how player B chooses his/her moves.

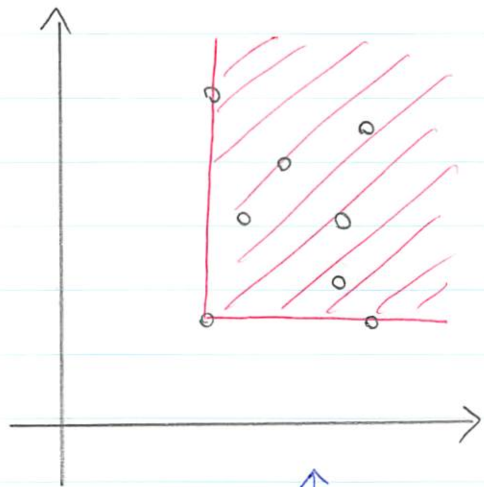
(Hironaka 1964)

3. Relation to sector decomposition

$$\int_0^1 dx_1 \cdots dx_n \; \frac{f(x_1,\cdots,x_n)}{P(x_1,\cdots,x_n)^{a+b\epsilon}}$$

reg.

entangled singularities at $x_i = 0$



After finite steps of sector decomp.,
we want to bring P to the form

$$\text{const.} \times x_1^{m_1} \cdots x_n^{m_n} \left[1 + \mathcal{O}(x_i) \right]$$

Each term of P : $\quad c\, x_1^{m_1} \cdots x_n^{m_n} \longleftrightarrow \vec{m} = (m_1, \cdots, m_n)$

Each point in M.

Sector decomposition by (x_i, x_j)

- sub-sectors

- $x_i > x_j \qquad x_j = x_i x_j'$

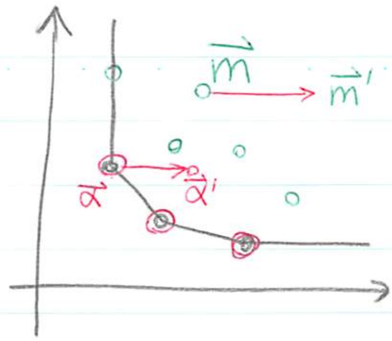
$$x_1^{m_1} \cdots x_n^{m_n} \longrightarrow x_1^{m_1} \cdots (x_i x_j')^{m_j} \cdots x_n^{m_n}$$

$$= x_1^{m_1} \cdots x_i^{m_i+m_j} \cdots x_j'^{m_j} \cdots x_n^{m_n}$$

$$(m_1, \cdots, \underset{\substack{\downarrow \\ m_i+m_j}}{m_i}, \cdots m_n)$$

- $x_j > x_i \qquad$ (similar)

General property of each move $\vec{m} \to \vec{m}'$



Def

$\underset{\vec{\alpha}\in}{M_c} (\subseteq M)$: the set of corners of Δ

Interior point $\vec{m} \in \underset{\not\in M_c}{M} \Rightarrow \vec{m}'$ is an interior point of M' (∗)

[Remains to be an interior pt. after a move.]

$\exists \underset{\vec{m}\neq}{\vec{\mu}} \in M$ s.t. $\vec{m} - \vec{\mu} \in \mathbb{R}^n_{\geq 0}$

move τ is a linear transf.

$$\begin{pmatrix} m_1 \\ \vdots \\ m_i \\ \vdots \\ m_n \end{pmatrix} \to \begin{pmatrix} m_1 \\ \vdots \\ m_i + m_j \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & \cdot & 1 \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_i \\ \vdots \\ m_n \end{pmatrix}$$

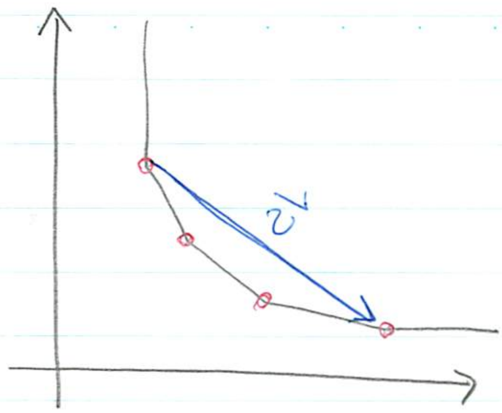
$$\underset{\substack{\cap \\ \mathbb{R}^n_{\geq 0}}}{\tau(\vec{m} - \vec{\mu})} = \tau(\vec{m}) - \tau(\vec{\mu}) = \vec{m}' - \vec{\mu}'$$

Thus, $M_c' \subseteq (M_c)' \equiv \tau(M_c)$ contraposition (対偶) of (∗)

Therefore, $\#(M_c') \leq \#(M_c)$

Good nature of the move, since we want to reduce the number of corners to one.

4. Zeillinger's algorithm



Let

$$V_M = \{ \vec{v} = \vec{\alpha}^{(a)} - \vec{\alpha}^{(b)} : \vec{\alpha}^{(a)}, \vec{\alpha}^{(b)} \in M_c \}$$

For $\forall \vec{v} \in V_M$

Length : $L(\vec{v}) \equiv \underset{\wedge}{\overset{0}{}}\max(v_i) - \underset{\vee}{\overset{0}{}}\min(v_i)$

Multiplicity : $N(\vec{v}) \equiv \# \{ j : v_j = \min(v_i) \} + \# \{ j : v_j = \max(v_i) \}$

the number of components of $\vec{v}$ equal to $\vec{v}$'s minimal or maximal component

e.g. $\vec{v} = (5, 0, -3)$ $L(\vec{v}) = 8$, $N(\vec{v}) = 2$

$\vec{v} = (1, -20, 1)$ $L(\vec{v}) = 21$, $N(\vec{v}) = 3$.

<u>Def</u>

Characteristic vector $\vec{v}_M$ of M is a minimal vector of V_M w.r.t. the inequality

$$(L(\vec{v}_M), N(\vec{v}_M)) \leq_{lex} (L(\vec{u}), N(\vec{u})), \quad \forall \vec{u} \in V_M$$

lexicographical ordering (辞書的順序)

There may be more than one characteristic vectors.

Winning strategy for Player A

Player A chooses $S = \{i, j\}$ s.t. v_i is a minimal and v_j is a maximal component of a characteristic vector of M.

e.g. if $\vec{v}_M = (5, 0, -3)$, $S = \{3, 1\}$.

Proof

$$(\#(M_c), L(\vec{v}_M), N(\vec{v}_M))$$

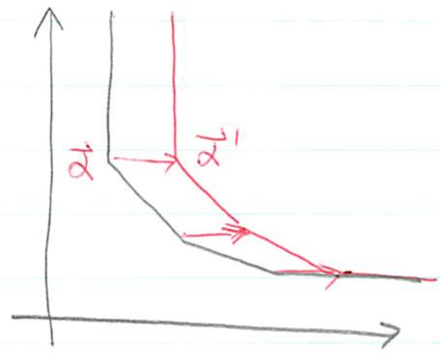
reduces monotonically w.r.t. lexicographical ordering.

i.e. $(\#(M'_c), L(\vec{v}_{M'}), N(\vec{v}_{M'})) <_{lex} (\#(M_c), L(\vec{v}_M), N(\vec{v}_M))$

$\#(M'_c) \leq \#(M_c)$, so suppose $\#(M'_c) = \#(M_c)$

In this case,

$$\vec{\alpha} \in M_c \Rightarrow \vec{\alpha}' = \tau(\vec{\alpha}) \in M'_c \qquad (\star)$$



Let $\vec{v}_M = (\underset{\text{minimal}}{v_1}, \underset{\text{maximal}}{v_2}, \cdots, v_n)$.

$$\tau(\vec{v}_M) = (v_1+v_2, v_2, \cdots, v_n) \quad \text{if B chooses } j=1$$
$$= (v_1, v_1+v_2, \cdots, v_n) \quad \text{" } j=2.$$

Since $v_1 < 0 < v_2$, $v_1 < v_1+v_2 < v_2$

$\therefore L(\tau(\vec{v}_M)) \leq L(\vec{v}_M)$ for both choices.

$\therefore L(\vec{v}_{M'}) \leq L(\underset{\substack{\cap \\ V_{M'}}}{\tau(\vec{v}_M)}) \leq L(\vec{v}_M)$

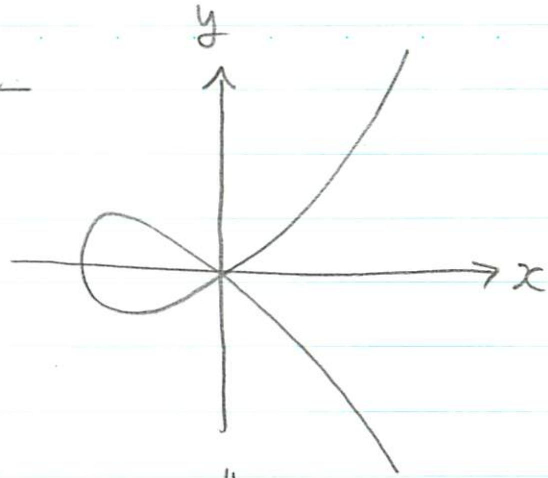
because of $(\star)$: $\tau(\vec{\alpha}_1 - \vec{\alpha}_2) = \tau(\vec{\alpha}_1) - \tau(\vec{\alpha}_2)$

If both equalities hold, there should be more than one maximal or minimal component of $\vec{v}_M$

$$N(\tau(\vec{v}_M)) < N(\vec{v}_M)$$
$$\rotatebox{90}{$\leq$}$$
$$N(\vec{v}_{M'})$$

Q.E.D.

5. Blowups of singularities, Visualization



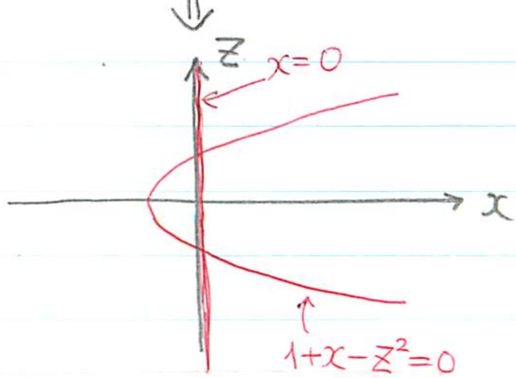
Example

$$x^3+x^2-y^2=0$$

In the subsector $x>y$:

$$y=xz \qquad \text{i.e.}\ z=y/x$$

$$x^3+x^2-(xz)^2 = x^2(1+x-z^2)=0$$



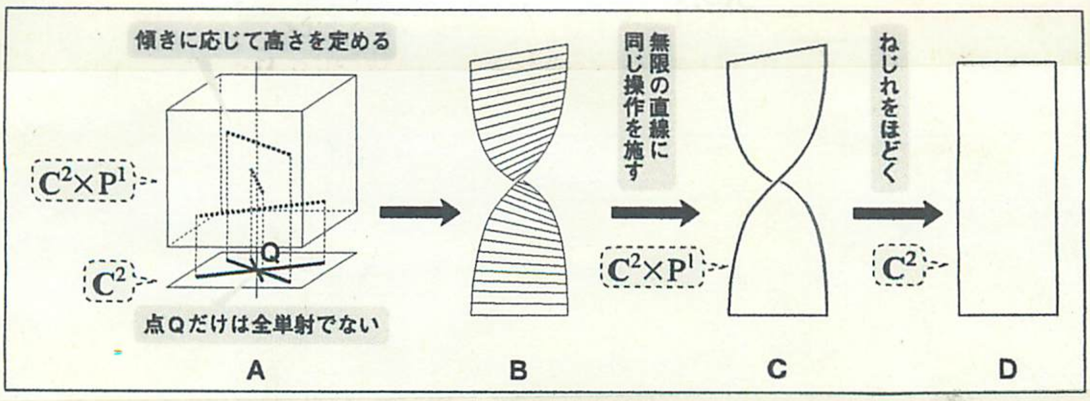


図3　土台作りの流れ

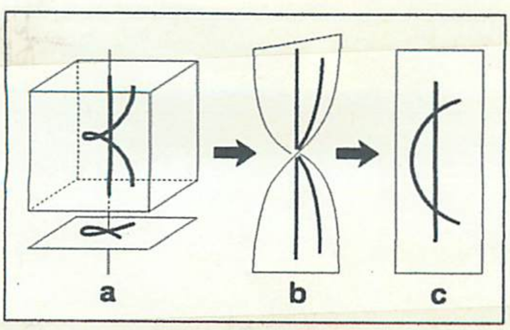


図4　特異点の解消