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# Ghost and gluon propagators in Yang-Mills theory with Gribov horizon and Kugo-Ojima color confinement criterion

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Based on

- Kugo-Ojima color confinement criterion and Gribov-Zwanziger horizon condition, arXiv:0904.4897[hep-th]. Phys.Lett.B. 678, 322-330 (2009).
- Infrared behavior of the ghost propagator in the Landau gauge Yang-Mills theory, arXiv:0907.3249 [hep-th]. Prog.Theor.Phys.**122**, No. 6 (2009), to be published.
- Decoupling and scaling solutions in Yang-Mills theory with the Gribov horizon, arXiv:0909.4866 [hep-th]
- A nilpotent "BRST" symmetry for the Gribov-Zwanziger theory, arXiv:0905.1899[hep-th].

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* On sabbatical leave of absence from Chiba University

# § Introduction

We consider the quantum Yang-Mills theory in the Landau gauge $\partial\mathscr{A} = 0$

$$Z_{\rm YM} := \int [d\mathscr{A}]\delta(\partial\mathscr{A})\det(-\partial D[\mathscr{A}]))\exp\{-S_{YM}[\mathscr{A}]\}. \tag{1}$$

However, the gauge fixing condition $\partial\mathscr{A} = 0$ can not fix the gauge uniquely. This is because each gauge orbit intersects the gauge fixing hypersurface $\Gamma := \{\mathscr{A}; \partial\mathscr{A} = 0\}$ many times. The unique representative can not be chosen. There are Gribov copies.

In order to avoid the Gribov copies, Gribov (1978) proposed to restrict the functional integral to the (1st) Gribov region $\Omega$

$$Z_{\rm YM} := \int_\Omega [d\mathscr{A}]\delta(\partial\mathscr{A})\det(-\partial D[\mathscr{A}]))\exp\{-S_{YM}[\mathscr{A}]\} \tag{2}$$

where

$$\Omega := \{\mathscr{A}; \partial\mathscr{A} = 0 \ \& \ -\partial D[\mathscr{A}] > 0\} \subset \Gamma \tag{3}$$

Note that $-\partial D[\mathscr{A} = 0] = -\partial\partial > 0$, i.e., $\{\mathscr{A} = 0\} \in \Omega$.
The boundary of $\Omega$ is called the Gribov horizon:

$$\partial\Omega := \{\mathscr{A}; \partial\mathscr{A} = 0 \ \& \ -\partial D[\mathscr{A}] = 0\} \tag{4}$$

He predicted that the resulting Green functions exhibit unexpected behavior in the deep infrared (IR) region and that they play the essential role in confinement.

Define the gluon 2-point function (full or complete propagator)

$$D_{\mu\nu}^{AB}(k) := \delta^{AB}\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{F(k^2)}{k^2} + \frac{\alpha}{k^2}\frac{k_\mu k_\nu}{k^2}\right] \quad (\alpha = 0) \tag{5}$$

and the ghost propagator

$$G^{AB}(k) := -\delta^{AB}\frac{G(k^2)}{k^2}. \tag{6}$$

The free case is

$$F(k^2) = 1, \quad G(k^2) = 1. \tag{7}$$

The UV behavior is given by one-loop resummed perturbation or one-loop RG

$$F(k^2) \sim \left(\ln\frac{k^2}{\Lambda^2}\right)^\gamma, \quad G(k^2) \sim \left(\ln\frac{k^2}{\Lambda^2}\right)^\delta, \quad \gamma = -\frac{13}{22}, \quad \delta = -\frac{9}{44}. \tag{8}$$

How about the IR behavior?

Mandelstam[1979] neglected ghost completely to yeild the linear potential?

$$\frac{F(k^2)}{k^2} \sim \frac{1}{k^4} \quad (k^2 \downarrow 0), \tag{9}$$

Gribov[1978] predicted their IR behaviors in the deep IR region $k^2 \ll 1$

$$\frac{F(k^2)}{k^2} \sim \frac{k^2}{(k^2)^2 + M^4} \downarrow 0, \quad \frac{G(k^2)}{k^2} \sim \frac{M^2}{(k^2)^2} \uparrow \infty \quad (k^2 \downarrow 0). \tag{10}$$

The gluon propagator vanishes in the IR limit $k^2 \downarrow 0$, while the ghost propagator becomes more singular than the free case in the IR region. This power like behavior should be compared with the UV behavior with the logarithmic corrections. ...

For the review [R. Alkofer and L. von Smekal, Phys.Rept.**353**, 281 (2001)]

$$F(k^2) = A \times (k^2)^{\alpha}, \quad G(k^2) = B \times (k^2)^{\beta}, \quad \alpha + 2\beta = 0, \quad 0 < A, B < \infty, \tag{11}$$

$$\alpha = 2\kappa > 1, \quad \beta = -\kappa < 0, \quad 1/2 < \kappa < 1 \text{ (Gribov } \kappa = 1).$$

Running coupling constant:

$$g^2(k) := g^2 F(k^2) G^2(k^2) \to 0 < g^2 A B^2 < \infty \ (k^2 \to 0) \text{ IR fixed point}$$

## The exact SD equations for gluon and ghost propagators in Gluodynamics: Diagrammatic representation

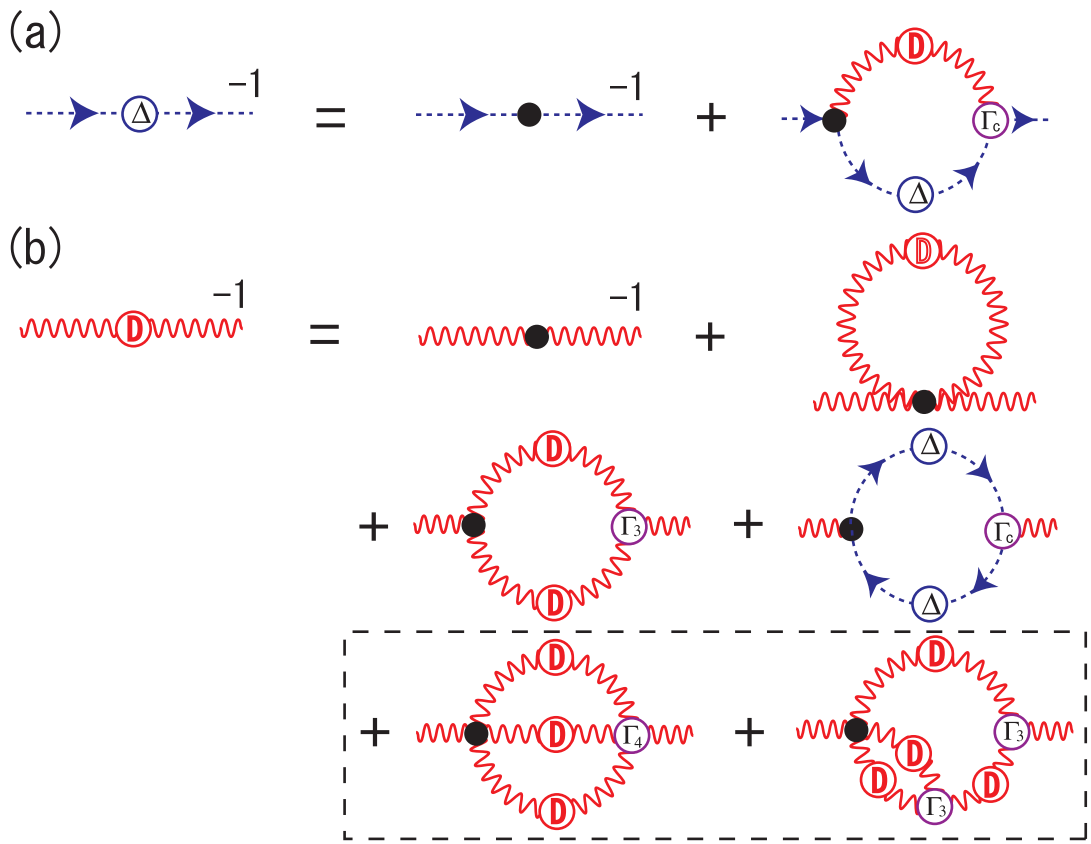


Figure 1: The coupled Schwinger–Dyson equations for the gluon and ghost propagators in Yang-Mills theory with the conventional Lorentz gauge fixing. (a) ghost equation, (b) gluon equation. Here $\Delta$ denotes the full ghost propagator and $D$ the full gluon propagator, while $\Gamma$ denote the four types of vertices. Two-loop diagrams are enclosed by the broken line.

# Comparison of the lattice data with the SDE results

◦ lattice data versus SDE results without 2-loop diagrams

[Fischer and Alkofer, hep-ph/0202202, PLB 536, 177 (2002).]

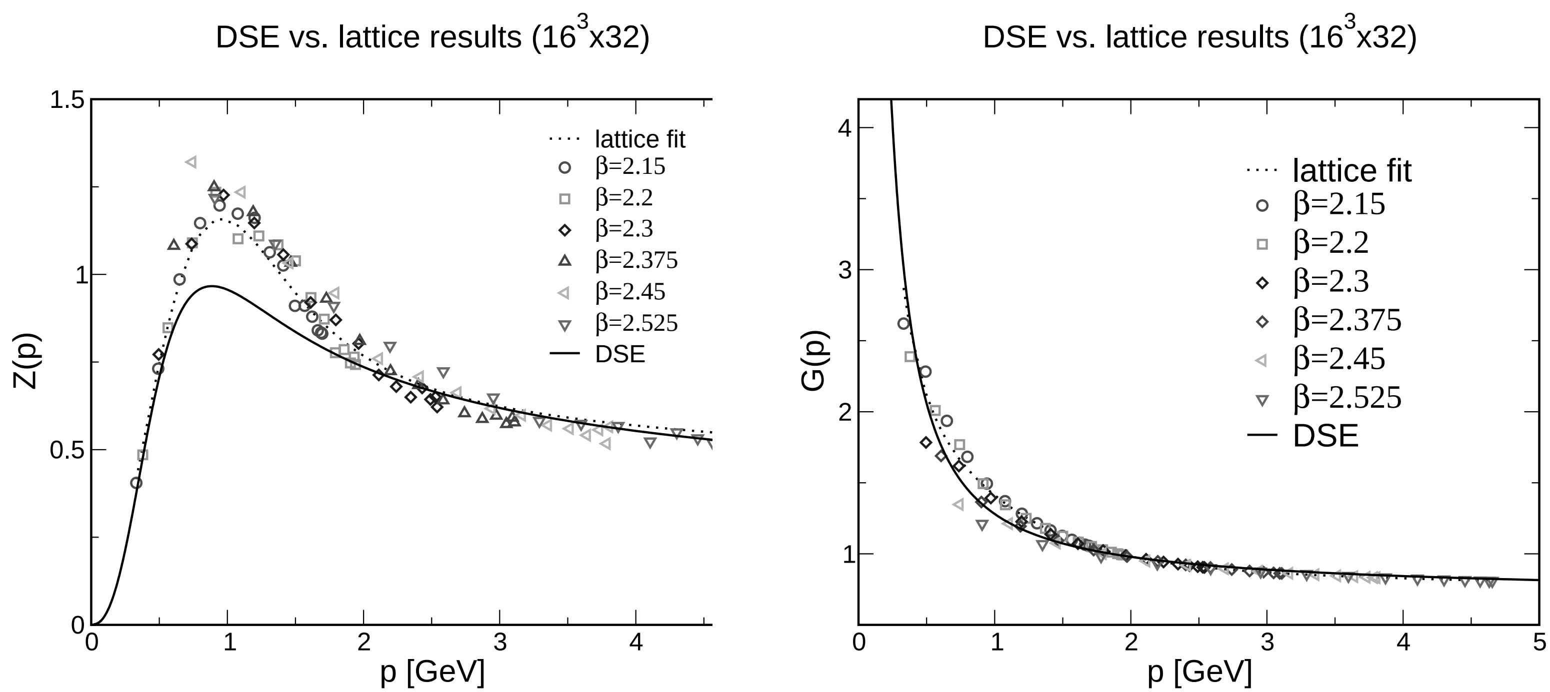


Figure 2: Solutions of the Dyson-Schwinger equations compared to recent lattice results for two colours [Langfeld:2002].

○ lattice data versus SDE results including all 2-loop diagrams [Bloch, hep-ph/0303125]

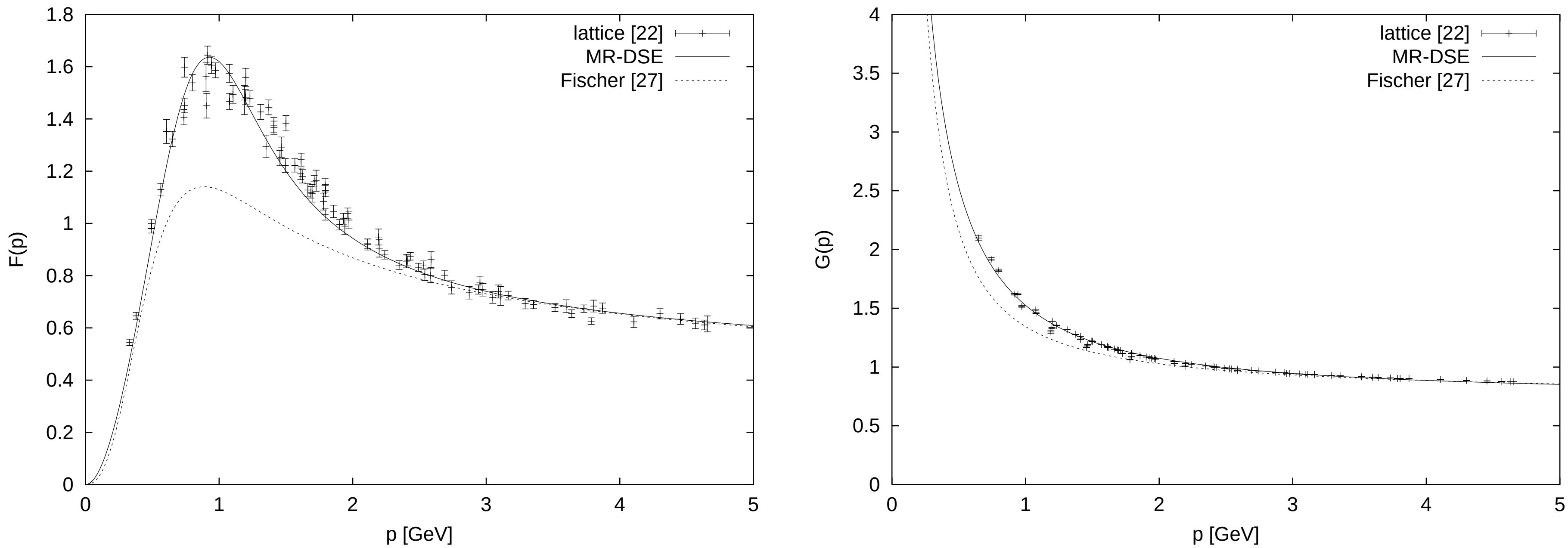


Figure 3: Comparison of the lattice data from ref.[2] with the DSE results using the two-loop improved MR truncation [3](with $\kappa = 0.5$) and from ref.[1]. *Upper pane:* Gluon dressing functions as function of momentum. *Lower pane:* Ghost dressing functions as function of momentum.

Two-loop diagrams neglected:
[1][Fischer and Alkofer, hep-ph/0202202]
Lattice simulation:
[2][Bloch, Cucchieri, Langfeld & Mendes, hep-lat/0209040]
Two-loop diagrams included:

[3][Bloch, hep-ph/0303125] [4][Alkofer, Fischer, Reinhardt and Smekal, hep-th/0304134]

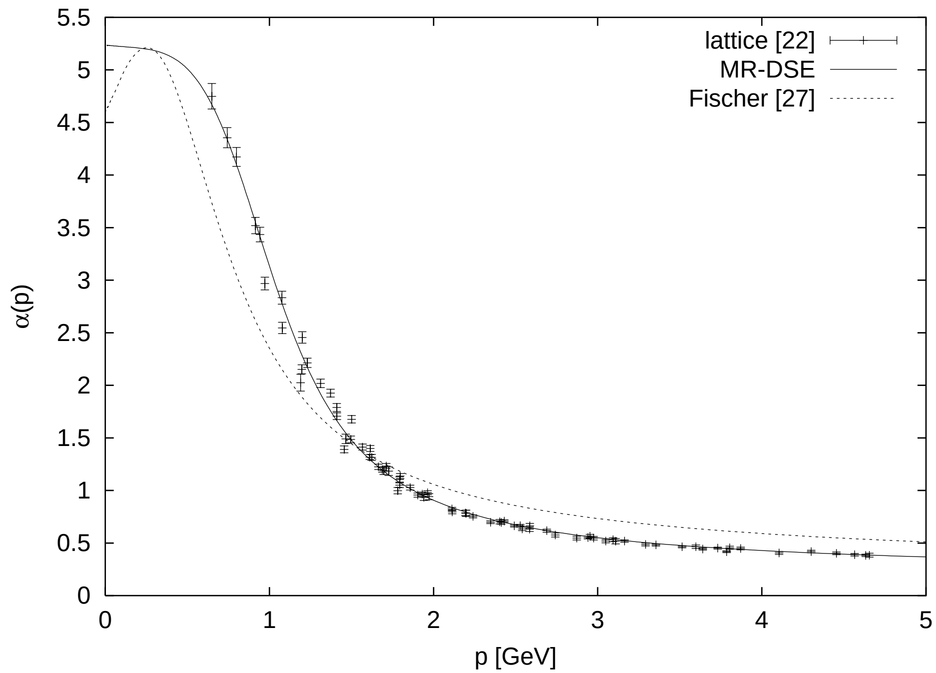


Figure 4: Running coupling as function of momentum for SU(2). Comparison of the lattice data from ref.[2] with the DSE results using the two-loop improved MR truncation [3](with $\kappa = 0.5$) and from ref.[1]. The fixed point of the MR truncation is $\alpha_0 = 5.24$.

◦ For the coupled renormalized SD equations for the quark, gluon and ghost propagators in the Landau gauge QCD (without two-loop digrams for the gluon equation) See [Fischer and Alkofer, hep-ph/0301094]

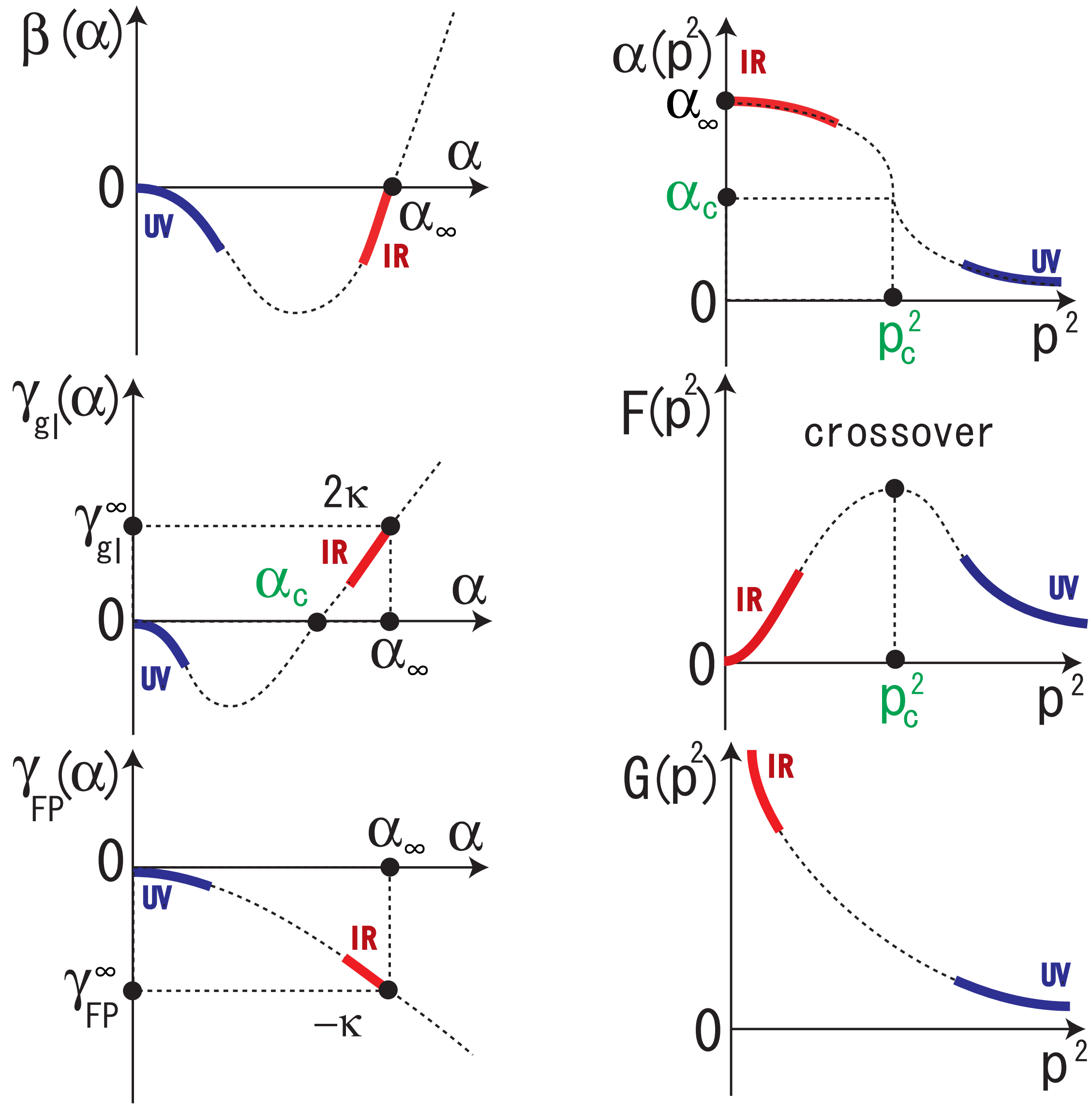


Figure 5: The schematic behavior of RG functions, running coupling constant and form factors for gluons and ghosts for Yang–Mills theory in the Lorentz–Landau gauge. (a) Beta function ↔ Running coupling, (b) Anomalous dim. of gluon ↔ Gluon form factor, (c) Anomalous dim. of ghost ↔ Ghost form factor.

This IR behavior was considered to be reasonable from the viewpoint of color confinement. Due to Kugo-Ojima (1977-1978), all color non-singlet objects can not be observed or confined, in other words, only color singlet objects are observed, if $u(0) = -1$ in the Lorentz covariant gauge (a sufficient condition for color confinement).

In the Landau gauge, Kugo-Ojima criterion for color confinement $u(0) = -1$ is equivalent to the divergent ghost dressing function $G(0) = \infty$, since in the Landau gauge

$$G(0) = [1 + u(0)]^{-1} \tag{12}$$

[Note that this relation is not exact, as shown in this talk.]

Until 2006, it seemed that this prediction has been confirmed by the Schwinger-Dyson equation (the scaling solution), the functional renormalization group equation and numerical simulations on lattice. $\Longrightarrow$ ghost dominance picture for confinement

So far so good.

However, ....

By careful analyses of the Schwinger-Dyson equation, socalled the decoupling solution was discovered:

$$F(k^2) = A' \times (k^2)^{\alpha}, \quad G(k^2) = B' \times (k^2)^{\beta}, \quad \alpha = 1, \quad \beta = 0, \quad 0 < A', B' < \infty,$$
$$g^2(k) := g^2 F(k^2) G^2(k^2) \cong g^2 A' B'^2 k^2 \to 0 \ (k^2 \to 0)$$

The ghost dressing function must be finite. See [Boucaud, Leroy, Yaouanc, Micheli, Pene and Rodriguez-Quintero, hep-ph/0803.2161, JHEP **06**, 099 (2008).]

Moreover, reexaminations of numerical simulations on lattices, functional renormalization group equation seem to converge the result:

The gluon propagator goes to the non-zero and finite constant in the IR limit, while the ghost propagator behaves like free (i.e., the ghost dressing function G(0) is non-zero and finite in the IR limit).

[Note that the Kugo-Ojima theory is based on the usual BRST formulation and does not take into account the Gribov problem where the exact color symmetry and the well-defined BRST charge are assumed.]

# § Main results of this talk

I discuss how the restriction of the integration region to the (1st) Gribov region constrains the possible value for the ghost dressing function and the Kugo-Ojima parameter for color confinement.

We analyse this issue within the Gribov-Zwanziger theory for the D-dimensional SU(N) Yang-Mills theory in the Landau gauge.

(1) I prove that the ghost dressing function $G(k^2)$ is non-zero finite in the limit $k \to 0$ and hence the ghost propagator behaves like free in the deep infrared regime.

(2) The Kugo-Ojima color confinement criterion u(0)=-1 is not satisfied in an original form. Rather, I find u(0)=-2/3 for $D = 4$ irrespective of $N$.

(3) However, it is possible to find a nilpotent "BRST" like symmetry in the Gribov-Zwanziger theory (restricted to the 1st Gribov region). $\boldsymbol{\delta} S_{\rm GZ} = \boldsymbol{\delta} \tilde{S}_\gamma \neq 0$
This is important to look for a modified color confinement criterion a la Kugo-Ojima.

These results are in harmony with decoupling solution of the Schwinger-Dyson equation, recent numerical simulation results on huge lattices.

However, they depend on the choice of the (non-local) horizon term.

## § Gribov-Zwanziger theory and horizon condition

- Gribov-Zwanziger theory [D.Zwanziger, Nucl. Phys. B**323**, 513–544 (1989)]

$$Z_\gamma := \int \mathcal{D}\mathscr{A}\,\delta(\partial^\mu \mathscr{A}_\mu)\det M \exp\{-S_{YM} - \gamma \int d^D x h(x)\}, \tag{1}$$

where $S_{YM}$ is the Yang-Mills action, $K$ is the Faddeev-Popov operator $K := -\partial_\mu D_\mu = -\partial_\mu(\partial_\mu + g\mathscr{A}_\mu \times)$ and $h(x) = h[\mathscr{A}](x)$ is the Zwanziger horizon function given by

$$h(x) := \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y). \tag{2}$$

Here the parameter $\gamma$ called the Gribov parameter is determined by solving a gap equation, commonly called the horizon condition:

$$\langle h(x) \rangle^\gamma = (N^2 - 1)D. \tag{3}$$

The action corresponding to the partition function (1) contains the *non-local* horizon term:

$$\int d^D x h(x) := \int d^D x \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y). \tag{4}$$

# § A localized Gribov-Zwanziger theory

[D. Zwanziger, Nucl. Phys. B**399**, 477–513 (1993).]

$$e^{-\gamma \int d^D x h(x)} = \int [d\xi][d\bar{\xi}][d\omega][d\bar{\omega}] \exp\left\{-\tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}]\right\}, \tag{1}$$

where

$$\begin{aligned}\tilde{S}_\gamma =: \int d^D x [&\bar{\xi}_\mu^{CA} K^{AB} \xi_\mu^{CB} - \bar{\omega}_\mu^{CA} K^{AB} \omega_\mu^{CB} \\ &+ i\gamma^{1/2} g f^{ABC} \mathscr{A}_\mu^B \xi_\mu^{AC} + i\gamma^{1/2} g f^{ABC} \mathscr{A}_\mu^B \bar{\xi}_\mu^{AC}].\end{aligned} \tag{2}$$

The localized action $S_{\rm GZ}$ for the Gribov-Zwanziger theory is obtained

$$\begin{aligned}S_{\rm GZ} =& S_{\rm YM}^{\rm tot}[\mathscr{A}, \mathscr{C}, \bar{\mathscr{C}}, \mathscr{B}] + \tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}] \\ =& S_{\rm YM}[\mathscr{A}] + S_{\rm GF+FP}[\mathscr{A}, \mathscr{C}, \bar{\mathscr{C}}, \mathscr{B}] + \tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}],\end{aligned} \tag{3}$$

where

$$\mathscr{L}_{\rm GF+FP} := \int d^D x \left\{\mathscr{B} \cdot \partial_\mu \mathscr{A}_\mu + \frac{\alpha}{2} \mathscr{B} \cdot \mathscr{B} + i\bar{\mathscr{C}} \cdot \partial_\mu D_\mu \mathscr{C}\right\}. \tag{4}$$

The localized GZ theory is known to be multiplicatively renormalizable to all orders.

$$\begin{aligned}\mathscr{A}_\mu =& Z_A^{1/2}\mathscr{A}_\mu^R, \quad \mathscr{B} = Z_B^{1/2}\mathscr{B}^R, \quad Z_B = Z_A^{-1}, \\ \mathscr{C} =& Z_C^{1/2}\mathscr{C}^R, \quad \bar{\mathscr{C}} = Z_C^{1/2}\bar{\mathscr{C}}^R, \\ g =& Z_g g_R, \quad Z_g = \tilde{Z}_1 Z_A^{-1/2} Z_C^{-1},\end{aligned} \tag{5}$$

$$\begin{aligned}\xi_\mu =& Z_\xi^{1/2}\xi_\mu^R, \quad \bar{\xi}_\mu = Z_{\bar{\xi}}^{1/2}\bar{\xi}_\mu^R, \quad Z_\xi = Z_{\bar{\xi}} = Z_C, \\ \omega_\mu =& Z_\omega^{1/2}\omega_\mu^R, \quad \bar{\omega}_\mu = Z_{\bar{\omega}}^{1/2}\bar{\omega}_\mu^R, \quad Z_\omega = Z_{\bar{\omega}} = Z_C, \\ \gamma =& Z_\gamma \gamma_R, \quad Z_\gamma = Z_A^{-1} Z_C^{-1},\end{aligned} \tag{6}$$

If we use the covariant derivative form, $\tilde{S}_\gamma = \int d^D x[\bar{\xi}_\mu^{CA} K^{AB} \xi_\mu^{CB} - \bar{\omega}_\mu^{CA} K^{AB} \omega_\mu^{CB} + i\gamma^{1/2} D_\mu[\mathscr{A}]^{AC} \xi_\mu^{AC} + i\gamma^{1/2} D_\mu[\mathscr{A}]^{AC} \bar{\xi}_\mu^{AC}]$. This means another non-local horizon term

$$h(x) = \int d^D y D[\mathscr{A}]_\mu^{AC}(x)(K^{-1})^{CE}(x,y) D[\mathscr{A}]_\mu^{AE}(y). \tag{7}$$

# § Horizon condition and ghost dressing function

[K.-I. K., arXiv:0904.4897[hep-th], arXiv:0907.3249 [hep-th]]:
The average of the horizon function is exactly rewritten as

$$\langle h(0) \rangle = -(N^2-1)\left\{ Du(0) + w(0) - G(0)[u(0)+w(0)]^2 \right\}, \tag{1}$$

where

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI} = \left[ g_{\mu\nu} u(k^2) + \frac{k_\mu k_\nu}{k^2} w(k^2) \right] \delta^{AB}. \tag{2}$$

Here $u(k^2)$ is the Kugo-Ojima function defined by

$$\langle (D_\mu \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k := \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{AB} u(k^2). \tag{3}$$

In the Landau gauge, the ghost dressing function $G(k^2)\delta^{AB} := -k^2 \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k$ satisfies

$$G(k^2) = [1 + u(k^2) + w(k^2)]^{-1}. \tag{4}$$

The last equality was derived by [Kugo, hep-th/9511033] and also in [P.A. Grassi, T. Hurth, A. Quadri, e-Print: hep-th/0405104, Phys.Rev. D**70**, 105014 (2004)].

The IR limit of the ghost dressing function satisfies ($G(0) \geq 0$)

$$G(0) = [1 + u(0) + w(0)]^{-1}. \tag{5}$$

The horizon condition is exactly rewritten as

$$\langle h(0) \rangle = -(N^2 - 1)\left\{Du(0) + w(0) - G(0)[u(0) + w(0)]^2\right\} = (N^2 - 1)D, \tag{6}$$

1) $G(0) = 0 \iff G(0)^{-1} = \infty \iff u(0) = \infty$ or $w(0) = \infty$ from (5)
$\Longrightarrow \langle h(0) \rangle = \infty$. The horizon condition (6) is not satisfied.

2) $G(0) = \infty \iff G(0)^{-1} = 0 \iff u(0) + w(0) = -1$ from (5)
$\Longrightarrow \langle h(0) \rangle = \infty$. The horizon condition (6) is not satisfied.

The horizon condition is satisfied only when $0 < G(0) < \infty$. The ghost propagator behaves like free in the deep IR regime.

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The Kugo-Ojima criterion $u(0) = -1 \iff G(0) = w(0)^{-1} \Longrightarrow$

$$\langle h(0) \rangle = -(N^2 - 1)\left\{-D + w(0) - [1/w(0)][-1 + w(0)]^2\right\} = (N^2 - 1)D$$

$\Longrightarrow w(0) = 1/2$ and $G(0) = 2$ for any $D$.
In other words, $w(0) \neq 1/2 \Longrightarrow u(0) \neq -1$ for any $D$.

## § Proof of the main result

Step 1: Rewriting the horizon condition

$$\langle h(x)\rangle = (\mathrm{dim}G)D \tag{1}$$

The average of the horizon function reads

$$\begin{aligned}\langle h(x)\rangle &= \int d^D y \langle g f^{ABC} \mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y)\rangle \\ &= -\int d^D y \langle g f^{ABC} \mathscr{A}_\mu^B(x) \mathscr{C}^C(x) \bar{\mathscr{C}}^E(y) g f^{AFE} \mathscr{A}_\mu^F(y)\rangle \\ &= -\int d^D y \langle (g\mathscr{A}_\mu \times \mathscr{C})^A(x)(g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A(y)\rangle \\ &= -\lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k .\end{aligned} \tag{2}$$

In what follows, we define the Fourier transform of the two-point function for composite operators by

$$\langle \phi_1^A \phi_2^B\rangle_k := \int d^D x e^{ik(x-y)}\langle 0|T[\phi_1^A(x)\phi_2^B(y)]|0\rangle . \tag{3}$$

Step 2: We consider the non-selfcontracted form

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k = \lambda^{AB}_{\mu\nu}(k) + \Delta^{AB}_{\mu\nu}(k), \tag{4}$$

where

$$\begin{aligned} \lambda^{AB}_{\mu\nu}(k) :=& \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{m1PI}}, \\ \Delta^{AB}_{\mu\nu}(k) :=& \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^C \rangle_k^{\mathrm{1PI}} \langle \mathscr{C}^C \bar{\mathscr{C}}^D \rangle_k \langle \mathscr{C}^D (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{1PI}}. \end{aligned} \tag{5}$$

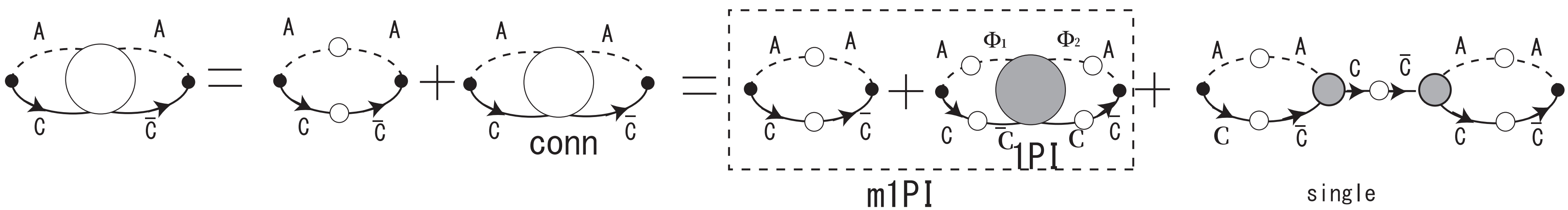



Figure 6: Diagrammatic representation of $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{conn}}$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{1PI}}$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{m1PI}}$.

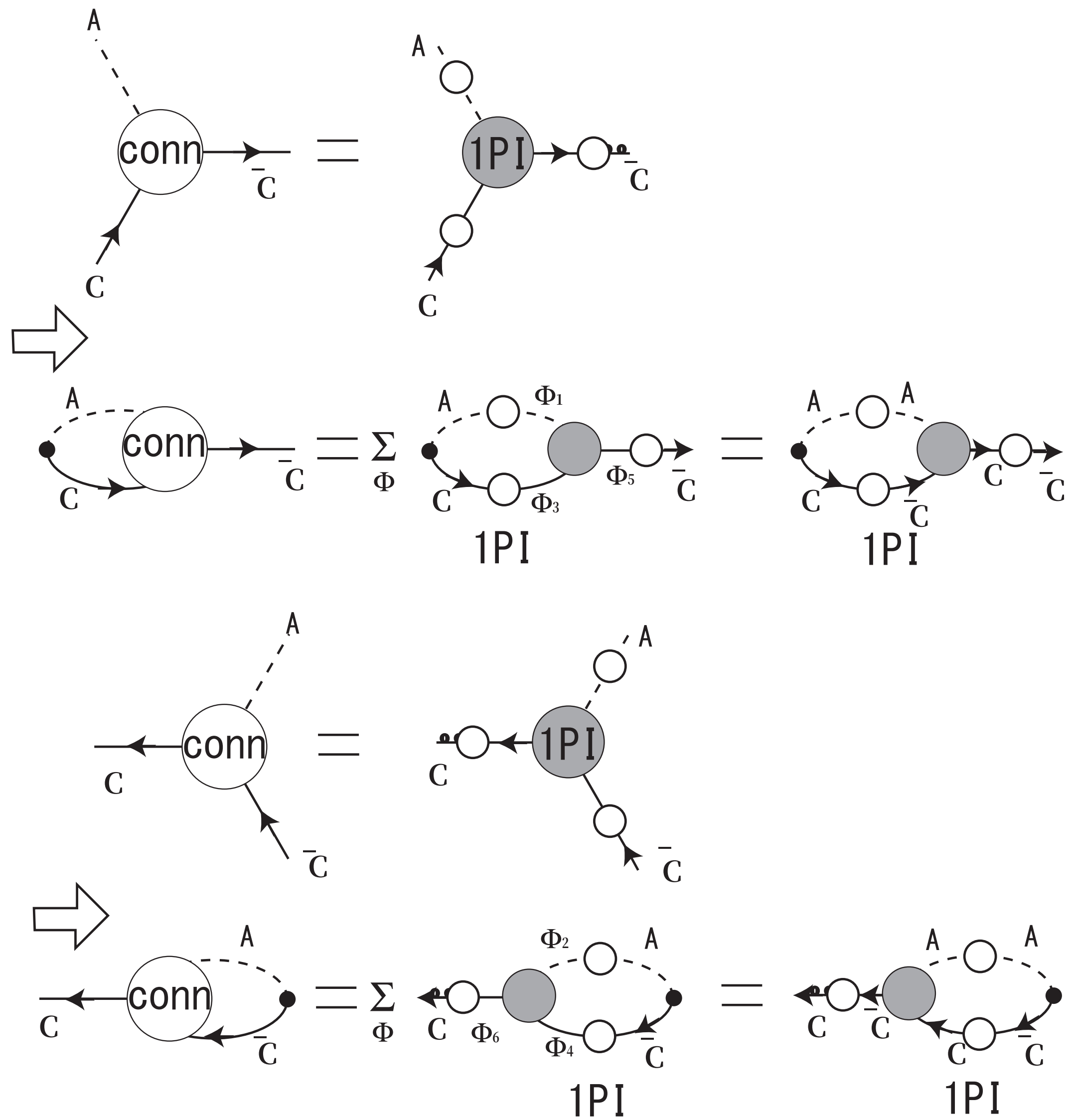



Figure 7: Diagrammatic representation of (a) $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{\rm 1PI}$, (b) $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$ and $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\rm 1PI}$.

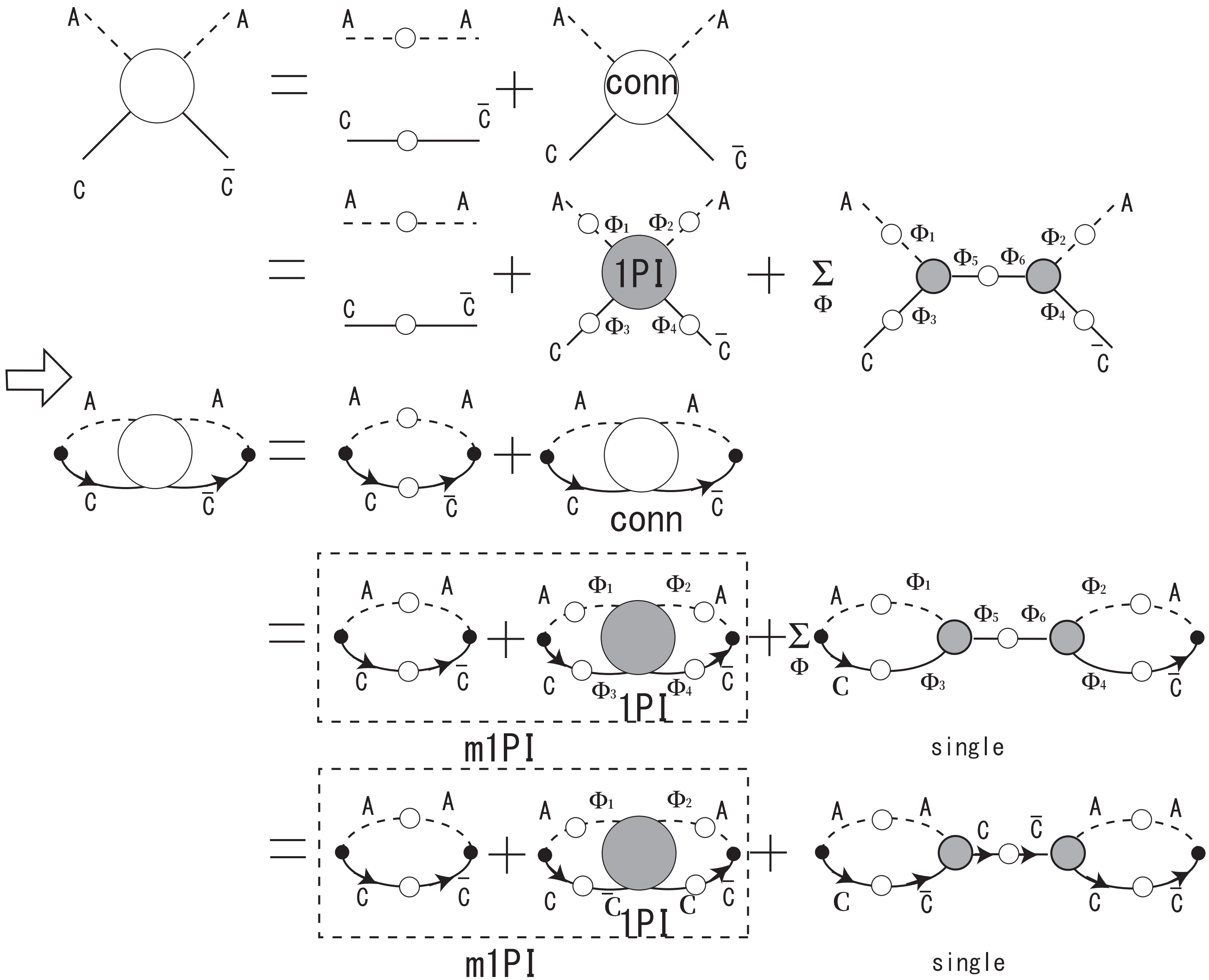



Figure 8: Diagrammatic representation of $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\text{conn}}$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\text{1PI}}$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\text{m1PI}}$.

Step 3: mutual relationships $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI}$, $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{1PI}$ (or $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}$) and $\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k$.

(a) In the manifestly covariant gauge of the Lorenz type,

$$ik_\mu \lambda_{\mu\nu}^{AB}(k) = \langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{1PI}, \tag{6}$$

(b) In the Landau gauge, the FP conjugation invariance leads to

$$-ik_\nu \lambda_{\mu\nu}^{AB}(k) = \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}, \tag{7}$$

(c)

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = -ik_\mu \left( -\delta^{AB} + \frac{-1}{k^2} \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} \right), \tag{8}$$

which is stronger than the resulting ghost propagator Schwinger-Dyson equation:

$$\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} = -k^2 \delta^{AB} - ik^\mu \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}. \tag{9}$$

Thus, $\lambda_{\mu\nu}^{AB}$ is related to the ghost propagator:

$$ik_\mu \lambda_{\mu\nu}^{AB}(k)(-ik_\nu) = ik^\mu \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = -\delta^{AB} k^2 - \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1}. \tag{10}$$

Step 4: The general form is (assuming unbroken color symmetry)

$$\lambda_{\mu\nu}^{AB}(k) = \left[ g_{\mu\nu} u(k^2) + \frac{k_\mu k_\nu}{k^2} w(k^2) \right] \delta^{AB}, \tag{11}$$

where $u$ is the Kugo-Ojima function (usually defined by)

$$\langle (D_\mu \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k := \delta^{AB} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) u(k^2), \tag{12}$$

and $w$ is an unfamiliar function.

In the Landau gauge, we obtain a relationship between the Kugo-Ojima function and the ghost dressing function defined by $G(k^2)\delta^{AB} := -k^2 \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k$

$$G(k^2)^{-1} = 1 + u(k^2) + w(k^2), \tag{13}$$

since

$$ik_\mu \lambda_{\mu\nu}^{AB}(k)(-ik_\nu) = -\delta^{AB} k^2 - \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1}. \tag{14}$$

In the Landau gauge, therefore, we find

$$\lambda^{AA}_{\mu\mu}(k) =(\mathrm{dim}G)[Du(k^2)+w(k^2)], \tag{15}$$

$$\begin{aligned}\Delta^{AA}_{\mu\mu}(k) =& - i\lambda^{AC}_{\mu\sigma}(k)k_\sigma \frac{-G(k^2)}{k^2}\delta^{CD} ik_\rho \lambda^{DA}_{\rho\mu}(k) \\ =& -(\mathrm{dim}G)G(k^2)[u(k^2)+w(k^2)]^2 \\ =& -(\mathrm{dim}G)\frac{[u(k^2)+w(k^2)]^2}{1+u(k^2)+w(k^2)}.\end{aligned} \tag{16}$$

The average of the horizon function reads

$$\begin{aligned}\langle h(0)\rangle =& -\lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k \\ =& -\lambda^{AA}_{\mu\mu}(0) - \Delta^{AA}_{\mu\mu}(0) \\ =& -(\mathrm{dim}G)\left\{Du(0)+w(0)-G(0)[u(0)+w(0)]^2\right\} \\ =& -(\mathrm{dim}G)\left\{Du(0)+w(0)-\frac{[u(0)+w(0)]^2}{1+u(0)+w(0)}\right\}.\end{aligned} \tag{17}$$

The existence of the last term $\Delta^{AA}_{\mu\mu}(0)$ is crucial to obtain a finite ghost dressing function at $k=0$.

## § Plugging into the Schwinger-Dyson equation

The Schwinger-Dyson (SD) equation for the ghost propagator is

$$\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} = -\delta^{AB} k^2 - i \frac{k^\mu}{k^2} \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}. \tag{1}$$

By using $-i\frac{k^\mu}{k^2}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = \frac{k^\mu k^\nu}{k^2}\frac{\lambda_{\mu\nu}^{AA}(k)}{(N^2-1)} = u(k^2) + w(k^2)$, the SD equation is rewritten as

$$G^{-1}(k^2) = 1 + u(k^2) + w(k^2). \tag{2}$$

Now we incorporate the horizon condition into the SD equation. Following the idea of Gribov, we substitute the horizon condition of the form

$$1 = \frac{\langle h(0) \rangle}{(N^2-1)D} = -u(0) - \frac{w(0)}{D} + \frac{G^{-1}(0) - 2 + G(0)}{D}. \tag{3}$$

Then, it is observed that the first term $-u(0)$ in the horizon condition cancels the term $u(k^2)$ at $k=0$ in the right-hand side of the SD equation,

$$G^{-1}(0) = \frac{G^{-1}(0) - 2 + G(0)}{D} + \left( -\frac{1}{D} + 1 \right) w(0). \tag{4}$$

By solving

$$G^2(0) - [2 + (1 - D)w(0)]G(0) + 1 - D = 0. \tag{5}$$

we have

$$G(0) = 1 + (1 - D)w(0)/2 + \sqrt{[1 + (1 - D)w(0)/2]^2 - 1 + D} > 0, \tag{6}$$

and $u(0) = -1 - w(0) + G^{-1}(0)$, i.e.,

$$u(0) = -1 - w(0) - \frac{1}{6}\{2 - 3w(0) - \sqrt{12 + [2 - 3w(0)]^2}\}. \tag{7}$$

This implies that the horizon condition determines the boundary value $G(0)$ in the ghost SD equation. Consequently, we have one-parameter family of solutions parameterized by $w(0)$.

We consider $G(0)$ and $u(0)$ as functions of $w(0)$. Both $G(0)$ and $u(0)$ are monotonically decreasing functions in $w(0)$; $G(0), u(0) \to \infty$ as $w(0) \to -\infty$, while $G(0) \to 0$ and $u(0) \to -5/3$ as $w(0) \to +\infty$. For $D = 4$, in particular, $G(0) = 3$ and $u(0) = -2/3$ at $w(0) = 0$; $G(0) = 2$ and $u(0) = -1$ at $w(0) = 1/2$.

Thus the scaling solution $G(0) = \infty$ is obtained only when $w(0) = -\infty$. Otherwise $w(0) > -\infty$, the decoupling solution $0 < G(0) < \infty$ is obtained.

Using a special value as an additional input $w(0) = 0$ by an assumption [Kugo95][Kondo09a] or by an independent argument [A.C. Aguilar, D. Binosi and J. Papavassiliou, arXiv:0907.0153 [hep-ph]], $G(0)$ is determined selfconsistently by solving the above SD equation as

$$G(0) = 1 + \sqrt{D} > 0, \quad u(0) = (-D \pm \sqrt{D})/(D-1). \tag{8}$$

In particular, for $D = 4$,

$$G(0) = 3 > 0, \quad u(0) = -2/3 \quad (D = 4). \tag{9}$$

The scaling solution $G(0) = \infty$ is obtained only when $\Delta^{AA}_{\mu\mu}(0)$ is vanishing. To obtain the scaling solution, the constant terms must cancel exactly or disappear at the $k = 0$ limit on the right-hand side of the SD equation. This is what implicitly assumed, but not stated explicitly, as pointed out by [Boucaud, Leroy, Yaouanc, Micheli, Pene and Rodriguez-Quintero, arXiv:0801.2721[hep-ph], JHEP **06**, 012 (2008).]

This result differs from the Gribov result. Is it possible to reconcile this result with the old Gribov result? Yes. See the following.

Recall that the Gribov result was obtained by taking into account the $O(g^2)$ terms.

The formal power series expansion in $u(0)$ yields the horizon condition

$$\langle h(0) \rangle = (N^2 - 1)\left\{-Du(0) + u(0)^2 - u(0)^3 + \cdots\right\} = (N^2 - 1)D. \qquad (10)$$

If we took into account only a linear term in $u(0) = O(g^2)$ on the left-hand side, then the horizon condition would lead to the Kugo-Ojima criterion $u(0) = -1$ and the ghost dressing function $G(0) = [1 + u(0)]^{-1}$ would diverge.

In this way we can reproduce the Gribov approximate (wrong?) result.

## § Another choice of the horizon term up to the total derivative term

We point out that the result crucially depends on the explicit form of the non-local horizon term adopted.

If the total derivative was neglected in the Gribov-Zwanziger theory, the horizon term could be rewritten as

$$\begin{aligned}\int d^D x h(x) &:= \int d^D x \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y) \\ ? &= \int d^D x \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y) \\ ? &= \int d^D x \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) D_\mu[\mathscr{A}]^{AE}(y), \qquad (1)\end{aligned}$$

The last horizon term yielded the average of the horizon function:

$$\langle h(0) \rangle = - \lim_{k \to 0} \langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k, \qquad (2)$$

Thus, the horizon condition for this horizon function defined from this form, i.e.,

$$\langle h(0)\rangle = -\lim_{k\to 0}\langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A\rangle_k = -(N^2-1)\{(D-1)u(0)-1\} = (N^2-1)D, \tag{3}$$

led to the Kugo-Ojima criterion:

$$u(0) = -1, \tag{4}$$

and the divergent ghost dressing function with an imput $w(0) = 0$:

$$G(0) = [1 + u(0) + w(0)]^{-1} = w(0)^{-1} = \infty. \tag{5}$$

Thus, if one starts from the horizon term in the last form of (1) by neglecting the total derivative term in the non-local horizon function, then one is led to the opposite conclusion to ours.

If we adopt another horizon function,

$$1 = \frac{\langle h(0) \rangle}{(N^2-1)D} = -\frac{\lim_{k \to 0} \langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k}{(N^2-1)D} = \frac{1}{D}[1+u(0)] - u(0). \tag{6}$$

the SD equation $G^{-1}(k^2) = 1 + u(k^2) + w(k^2)$ is rewritten as

$$G^{-1}(k^2) = \frac{1}{D}[1+u(0)] - u(0) + u(k^2) + w(k^2). \tag{7}$$

In the deep IR limit, such a cancellation occurs for $u(0)$:

$$G^{-1}(0) = \frac{1}{D}[1+u(0)] + w(0) \Longrightarrow G^{-1}(0) = \frac{1}{D}G^{-1}(0) - \frac{1}{D}w(0) + w(0) \tag{8}$$

This is solved for $D \neq 1$ to give

$$G^{-1}(0) = w(0). \tag{9}$$

and

$$u(0) = -1. \tag{10}$$

Thus, we have one-parameter family of solutions parameterized by $w(0)$. The scaling solution $G(0) = \infty$ is obtained only when $w(0) = 0$. Otherwise $0 < w(0) < \infty$, the decoupling solution $\infty > G(0) > 0$ is obtained. It should be remarked that the Kugo-Ojima condition $u(0) = -1$ is always satisfied.

However, this does not immediately mean the enhancement of the ghost propagator, contrary to the usual claim found in literatures.

If we require that two horizon conditions give the same result, then the relation

$$u(0) = -1/2 - w(0) \tag{11}$$

must be satisfied for any $D$. This implies that the common solution is found for any $D$

$$G(0) = 2, \quad u(0) = -1, \quad w(0) = \frac{1}{2}. \tag{12}$$

# § Lattice data for the Kugo-Ojima parameter

• H. Nakajima and S. Furui, hep-lat/9909008,
• H. Nakajima and S. Furui, hep-lat/0006002,

• A. Sternbeck, section 5.2 in hep-lat/0609016. [Ph.D. thesis]
• A. Sternbeck, E.-M. Ilgenfritz, M. Müller-Preussker, A. Schiller and I.L. Bogolubsky, hep-lat/0610053,

In pure Yang-Mills theory,

$$u(0) = -0.6 \sim -0.8 > -1$$

See Figure 5.3 in section 5.2.2 on page 104

$$\tilde{u}(k^2, \mu^2) := Z^{-1}(k^2, \mu^2) - 1 = u(k^2, \mu^2) + w(k^2)$$

See Figure 5.4 on page 106 [Figure 6 of hep-lat/0610053],

$$u(k^2, \mu^2), \quad \tilde{u}(k^2, \mu^2) - u(k^2, \mu^2) = w(k^2)$$

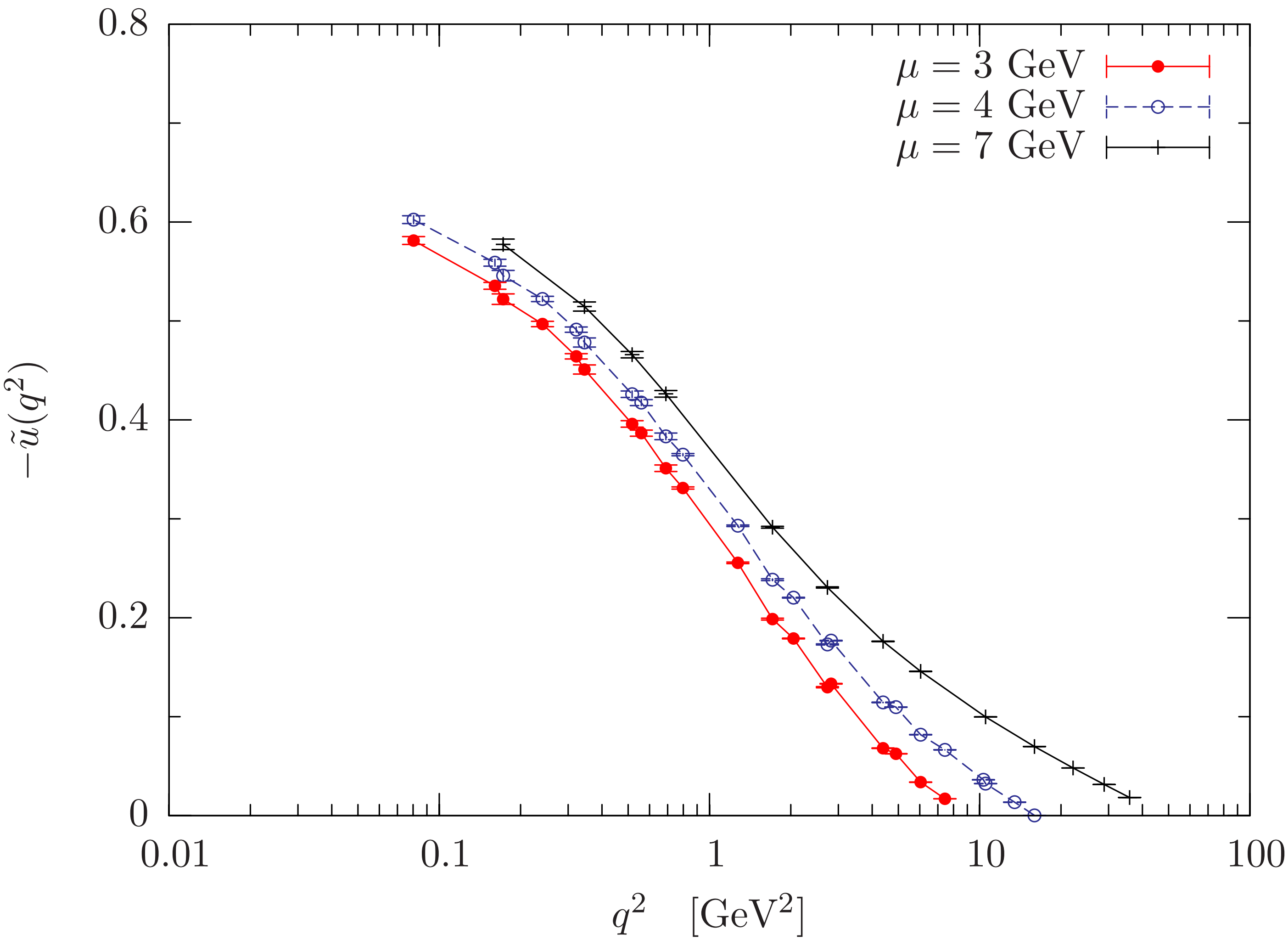
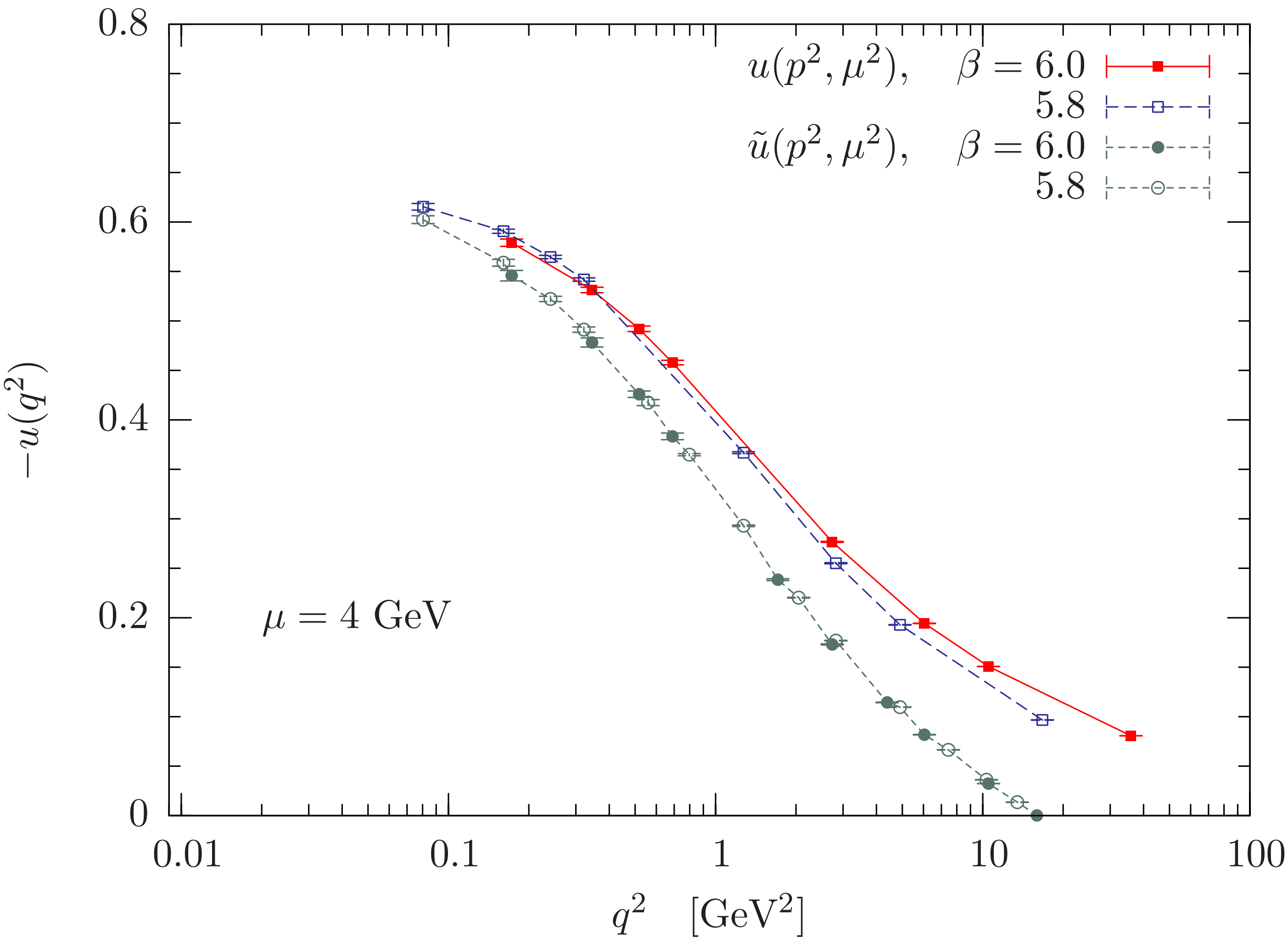



Figure 9: The asymptote $-\tilde{u}(q^2, \mu^2)$ as defined in ( ) is shown as a function of momentum $q^2$. For the ghost dressing function we used our data at $\beta = 5.8$ and $6.0$ renormalized either at $\mu = 3$, $4$ or $7$ $GeV$. The lattice size is $32^4$. Lines are drawn to guide the eye.




Data for the function $u(q^2, \mu^2)$ at $\beta = 5.8$ and 6.0 are shown using full and open squares. Additionally, data of the asymptote $\tilde{u}(q^2, \mu^2)$ are shown at the same $\beta$ values (circles). All data refer to the same quenched configurations on a $32^4$ lattice and are renormalized at $\mu = 4\ GeV$ as described in the text. Lines are drawn to guide the eye.

# § Lattice data for ghost and gluon dressing function

[Bogolubsky-Ilgenfritz-Muller-Preussker-Sternbeck-2009-PLB676-69-73, 0901.0736]
SU(3)

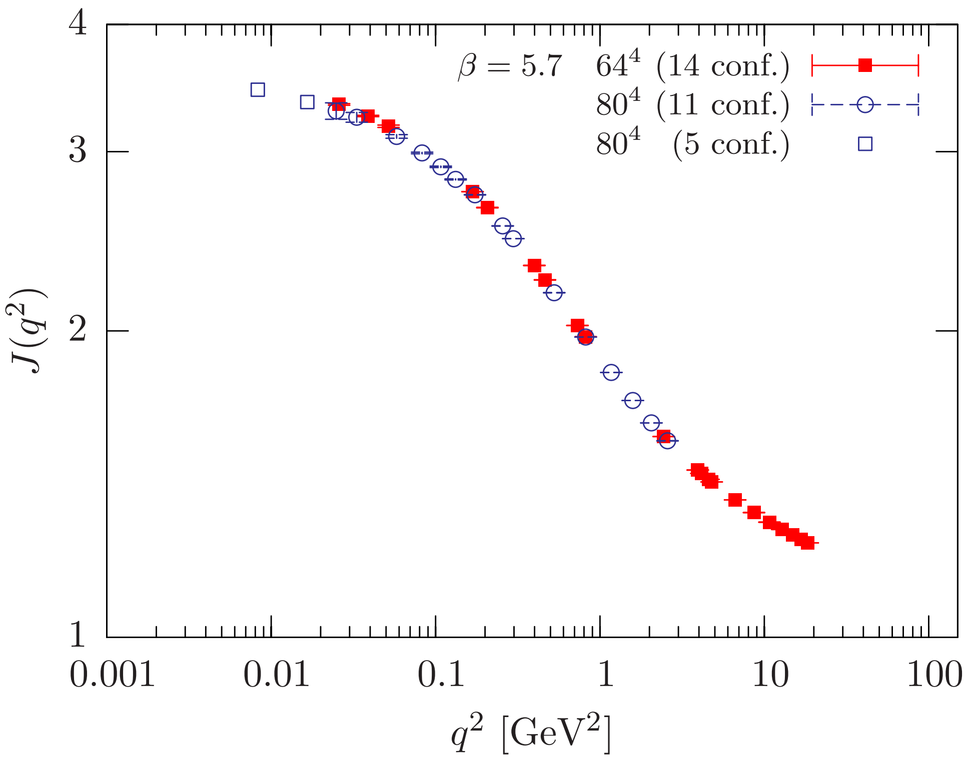



Figure 10: [SU(3)] Bare ghost dressing function $J(q^2)$ versus $q^2$ for $L = 64, 80$ at $\beta = 5.70$. Errors are not shown at the two lowest $q^2$ (squares).

[Bogolubsky-Ilgenfritz-Muller-Preussker-Sternbeck, 0812.2761] SU(2)

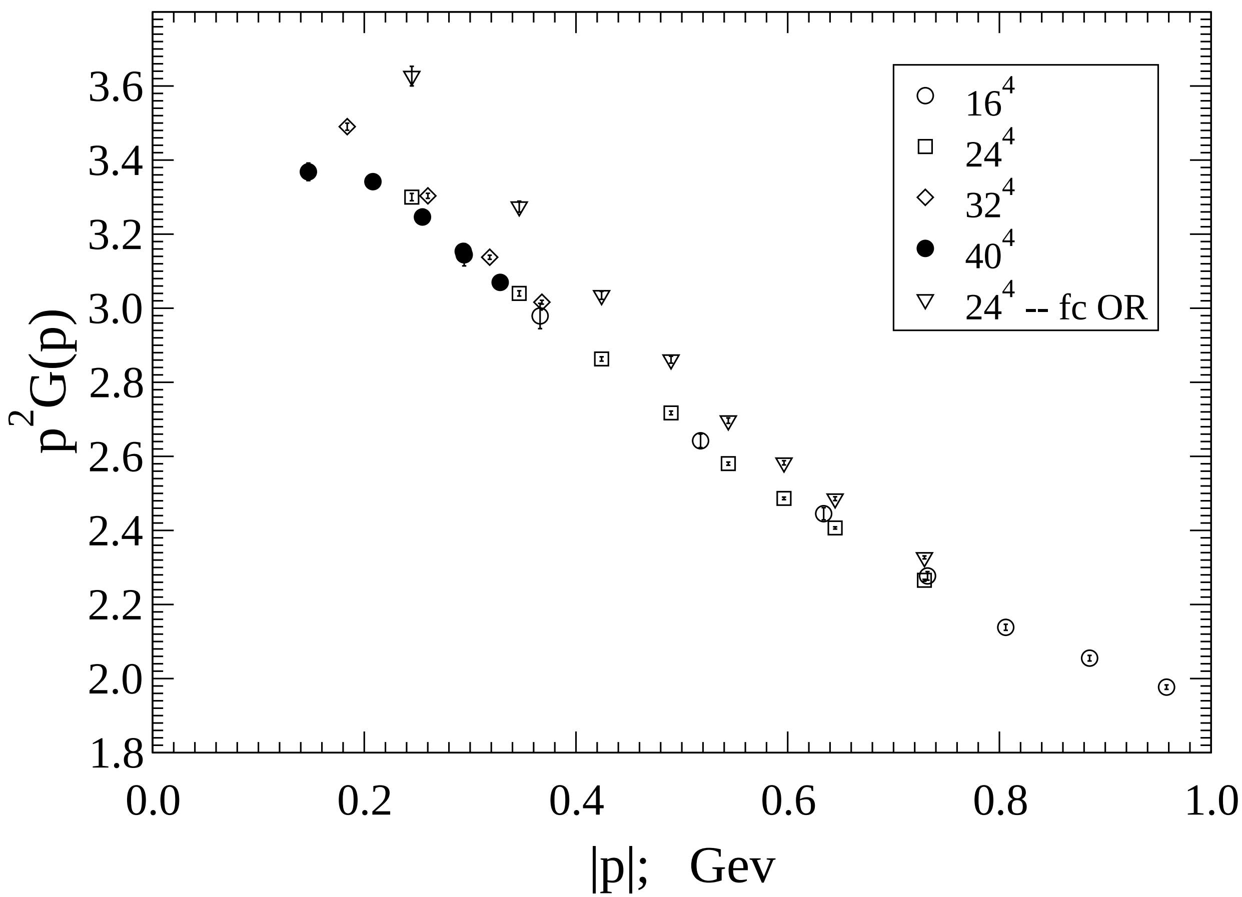



Figure 11: [SU(2)] The momentum dependence of the ghost dressing function $p^2 \cdot G(p)$ on the various lattices. For comparison results obtained with OR algorithm on $24^2$ lattices are also shown.

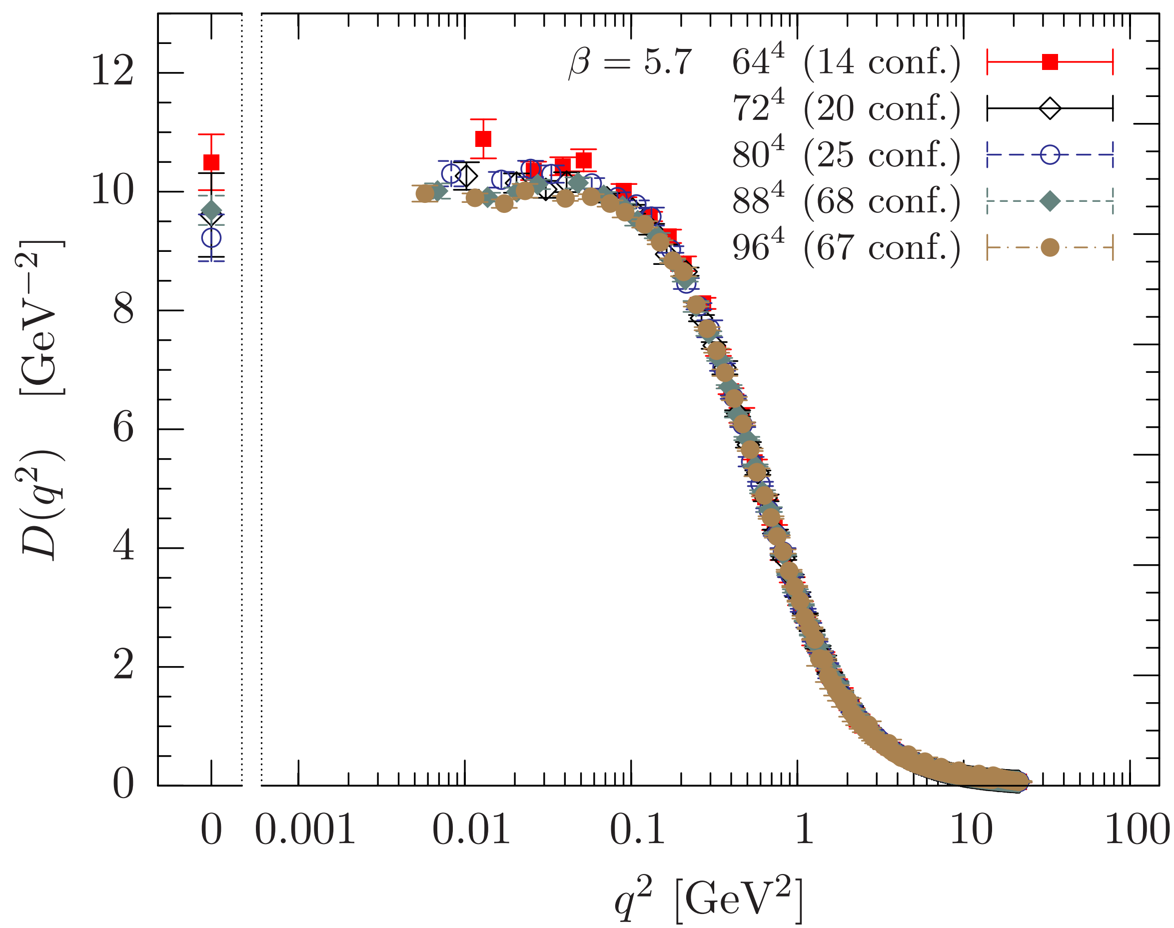



Figure 12: [SU(3)] The bare lattice gluon propagator $D(q^2)$ versus $q^2$ for $\beta = 5.70$ and various lattice sizes. We also show data on $D(0)$ (left).

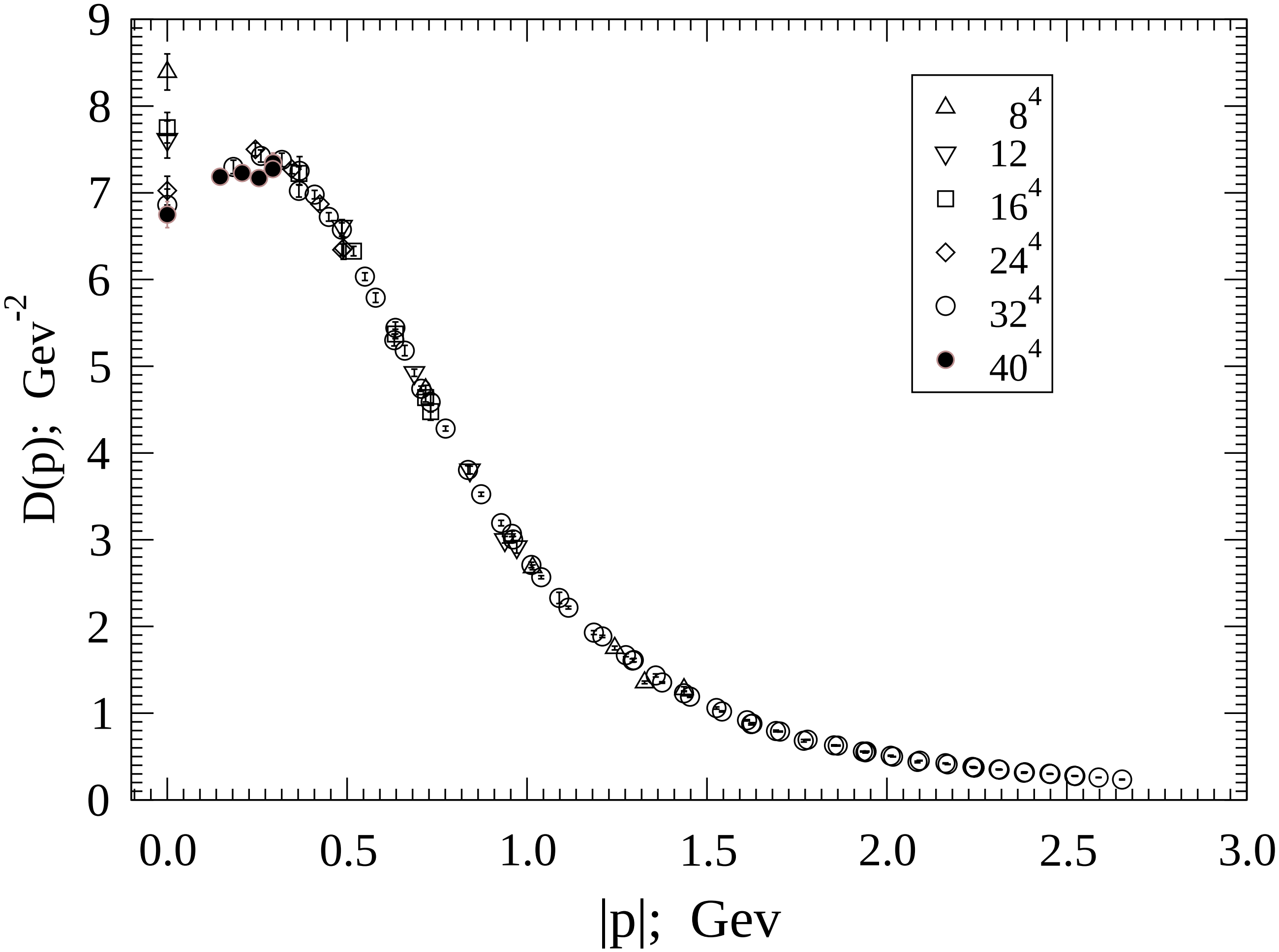



Figure 13: [SU(2)] The momentum dependence of the gluon propagator $D(p)$ on various lattice size. bc results are shown throughout.

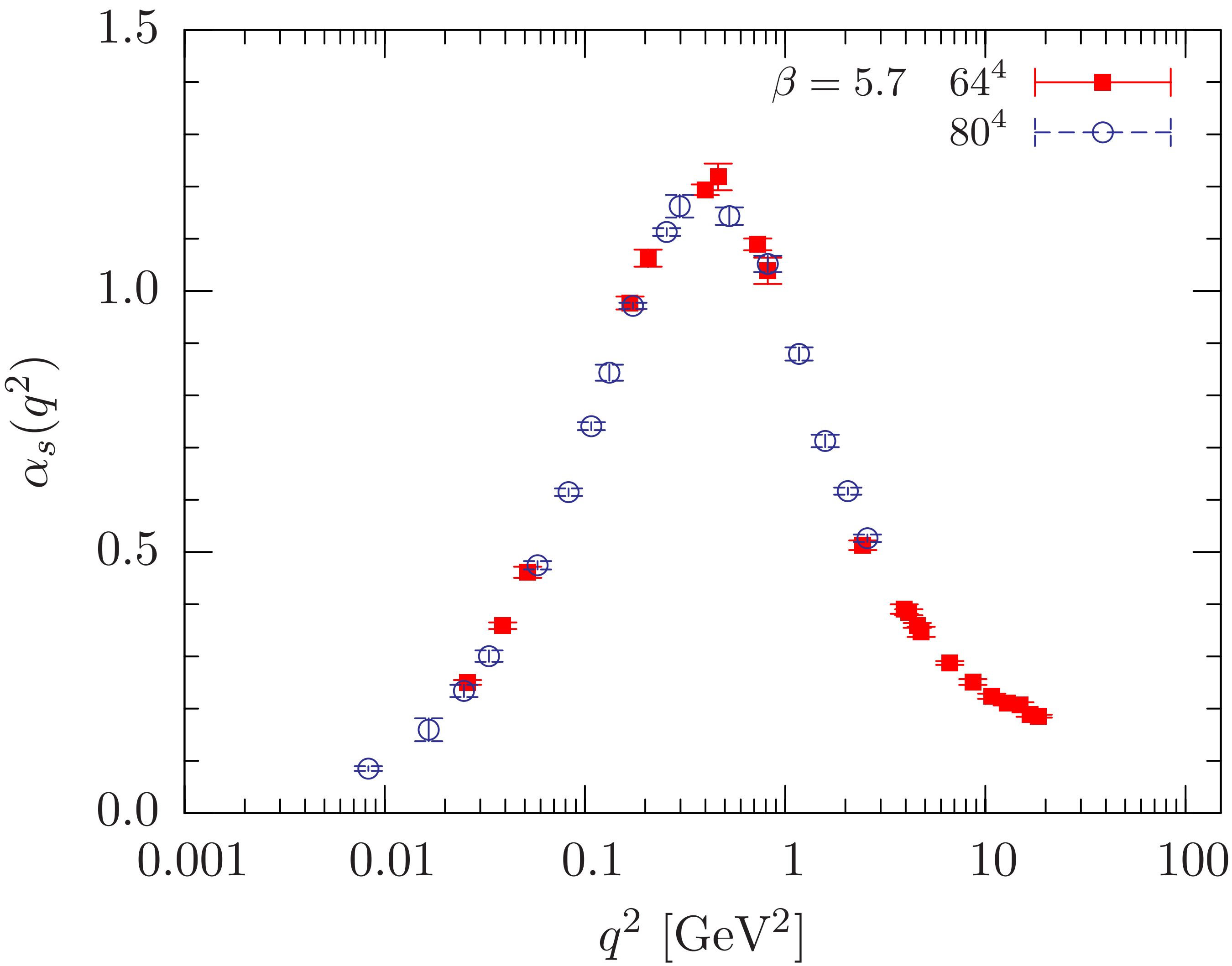



Figure 14: [SU(3)] Running coupling $\alpha_s(q^2)$ versus $q^2$ for lattice sizes $64^4$ and $80^4$ at $\beta = 5.70$.

# § Remark: A diagram connected by a single ghost line

Comparison with the paper: D. Zwanziger, Some exact infrared properties of gluon and ghost propagators and long-range force in QCD, arXiv:0904.2380[hep-th].

The essential difference between the Zwanziger result and ours comes from the 2nd term $\Delta_{sing}(k) := \Delta^{AA}_{\mu\mu}(k)$.

In fact, if $\Delta_{sing}(k) := \Delta^{AA}_{\mu\mu}(k)$ vanishes in the limit $k = 0$ as claimed in the paper, the average of the horizon function became instead equal to

$$\langle h(0) \rangle = -\lambda^{AA}_{\mu\mu}(0) = -(N^2 - 1)Du(0), \tag{1}$$

and the horizon condition $\langle h(0) \rangle = (N^2 - 1)D$ led to the Kugo-Ojima criterion and the divergent ghost dressing function:

$$u(0) = -1, \quad G(0) = [1 + u(0) + w(0)]^{-1} = w(0)^{-1} = \infty. \tag{2}$$

However, $\Delta_{sing}(k)$ does not vanish and remains non-zero even after taking the $k = 0$ limit, as we have examined.

## § $u(0)$ and $w(0)$ as functions of $G(0)$ [Boucaud etal., 0909.2615]

For the first horizon term $\frac{\langle h(0)\rangle}{D(N^2-1)} = D^{-1}\{-(D-1)u(0) - G(0)[u(0)+w(0)]\}$

$$
\begin{aligned}
u_\Lambda(0) &= \frac{1}{D-1}\left\{G_\Lambda(0) - 1 - D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\} \\
w_\Lambda(0) &= \frac{1}{D-1}\left\{-G_\Lambda(0) + 2 - D + D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\} + \frac{1}{G_\Lambda(0)}
\end{aligned}
\tag{1}
$$

Using the horizon condition,

$$
\begin{aligned}
u_\Lambda(0) &= \frac{1}{D-1}\{G_\Lambda(0) - 1 - D\} \\
w_\Lambda(0) &= \frac{1}{D-1}\{-G_\Lambda(0) + 2\} + \frac{1}{G_\Lambda(0)}
\end{aligned}
\tag{2}
$$

If $G_\Lambda(0) \to \infty$ as $\Lambda \to \infty$, then $u_\Lambda(0) \to +\infty$ and $w_\Lambda(0) \to -\infty$ such that $u_\Lambda(0) + w_\Lambda(0) \to -1$.

[Boucaud etal., 0909.2615]

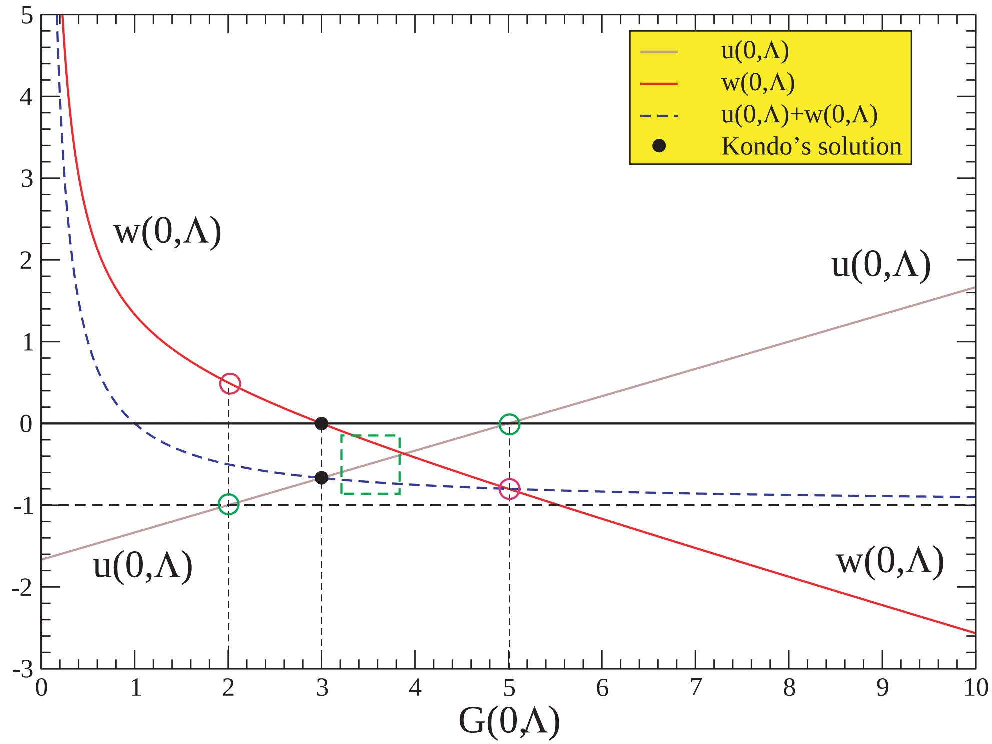



Figure 15: The solutions for $u_\Lambda(0)$ and $w_\Lambda(0)$ plotted as a function of $G_\Lambda(0)$. The particular solution proposed by Kondo (black circles), obtained by imposing $w(0,\Lambda) = 0$, corresponds to the intersection of $u + w$ and $u$. The current lattice solutions for the bare ghost dressing functions at vanishing momentum lie inside the green dotted square.

Remark: If $\Delta$ is neglected, then $\frac{\langle h(0)\rangle}{D(N^2-1)} = D^{-1}\{-Du(0) - w(0)\}$

$$u_\Lambda(0) = \frac{1}{D-1}\left\{1 - \frac{1}{G_\Lambda(0)} - D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\}$$
$$w_\Lambda(0) = \frac{1}{D-1}\left\{-D + D\frac{1}{G_\Lambda(0)} + D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\} \tag{3}$$

Using the horizon condition,

$$u_\Lambda(0) = \frac{1}{D-1}\left\{1 - \frac{1}{G_\Lambda(0)} - D\right\} < -1$$
$$w_\Lambda(0) = \frac{D}{D-1}\frac{1}{G_\Lambda(0)} \tag{4}$$

If $G_\Lambda(0) \to \infty$ as $\Lambda \to \infty$, then $u_\Lambda(0) \to -1$ and $w_\Lambda(0) \to 0$ such that $u_\Lambda(0) + w_\Lambda(0) \to -1$. However, $u_\Lambda(0) < -1$ which contradicts with the lattice result.

For the second horizon term $\frac{\langle h(0)\rangle}{D(N^2-1)} = D^{-1}\{-(D-1)u(0)+1\}$

$$u_\Lambda(0) = \frac{1}{D-1}\left\{1 - D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\}$$
$$w_\Lambda(0) = -\frac{1}{D-1}\left\{D - D\left[\frac{\langle h(0)\rangle}{D(N^2-1)}\right]\right\} + \frac{1}{G_\Lambda(0)} \tag{5}$$

Using the horizon condition,

$$u_\Lambda(0) = -1 \ (\Lambda - \text{indep.})$$
$$w_\Lambda(0) = \frac{1}{G_\Lambda(0)} \tag{6}$$

$u_\Lambda(0) = -1$ independently of $\Lambda$. If $G_\Lambda(0) \to \infty$ as $\Lambda \to \infty$, then $w_\Lambda(0) \to 0$ such that $u_\Lambda(0) + w_\Lambda(0) \to -1$.

## § Removing ultraviolet divergence and renormalization

We have calculated the horizon condition using

$$\langle h(x) \rangle = - \lim_{k \to 0} \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k \tag{1}$$

However, the composite operator $(g\mathscr{A}_\mu \times \mathscr{C})$ is not multiplicative renormalizable due to operator mixing:

$$g\mathscr{A}_\mu \times \mathscr{C} = Z_C^{-1/2} (g\mathscr{A}_\mu \times \mathscr{C})_R + Z_C^{-1/2} (1 - Z_C) \partial_\mu \mathscr{C}_R. \tag{2}$$

On the other hand, the composite operator$D_\mu[\mathscr{A}]\mathscr{C}$ is multiplicative renormalizable

$$D_\mu[\mathscr{A}]\mathscr{C} = Z_C^{-1/2} (D_\mu[\mathscr{A}]\mathscr{C})_R, \tag{3}$$

In this sense, the horzon condition is good:

$$\langle h(x) \rangle = - \lim_{k \to 0} \langle (D_\mu[\mathscr{A}]\mathscr{C})^A (D_\mu[\mathscr{A}]\bar{\mathscr{C}})^A \rangle_k, \tag{4}$$

The SD equation for the ghost propagator is form-invariant under the renormalization:

$$\langle \mathscr{C}_R^A \bar{\mathscr{C}}_R^B \rangle_k = -\delta^{AB} \frac{1}{k^2} - i \frac{k^\mu}{k^2} \langle (g\mathscr{A}_\mu \times \mathscr{C})_R^A \bar{\mathscr{C}}_R^B \rangle_k. \tag{5}$$

An unrenormalized horizon condition $0 = (N^2 - 1)D - \langle h(0) \rangle$ is

$$\begin{aligned} 0 =& (N^2 - 1)D - \langle h(0) \rangle = (N^2 - 1)D + \lim_{k \to 0} \langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k \\ =& (N^2 - 1)(D - 1)[1 + u(0)], \end{aligned} \tag{6}$$

The multiplicatively renormalized horizon condition is

$$\begin{aligned} 0 =& Z_C[(N^2 - 1)D - \langle h(0) \rangle] = (N^2 - 1)(D - 1) Z_C [1 + u(0)] \\ =& (N^2 - 1)(D - 1)[1 + u_R(0)], \end{aligned} \tag{7}$$

if we adopt the renormalization

$$1 + u(0) = Z_C^{-1} [1 + u_R(0)]. \tag{8}$$

The horizon condition is satisfield only when $u_R(0) = -1$ and $G_R^{-1}(0) = w_R(0)$ also after normalization. We have a definite result.

We return to the first horizon condition:

$$\langle h(x) \rangle = -\lim_{k\to 0} \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k.$$

<u>The conventional method:</u> We must use the localized GZ theory which is multiplicative renormalizable. The Slavnov-Taylor identity means

$$\langle i\gamma^{-1/2} g^2 f^{ABC} \mathscr{A}_\mu^B(x) \bar{\xi}_\mu^{CA}(x) \rangle = \langle h(x) \rangle = (N^2-1)D. \tag{9}$$

This is rewritten into the covariant derivative form, since the average does not depend on $x$ due to the translational invariance:

$$\langle i\gamma^{-1/2} g D_\mu[\mathscr{A}]^{AC} \bar{\xi}_\mu^{CA} \rangle = \langle h(0) \rangle = (N^2-1)D. \tag{10}$$

Then the horizon condition is multiplicatively renormalized:

$$\langle i\gamma_R^{-1/2} g_R (D_\mu[\mathscr{A}]^{AC} \bar{\xi}_\mu^{CA})_R \rangle = Z_C \langle h(0) \rangle = Z_C (N^2-1)D. \tag{11}$$

If the horizon condition is incorporated into the SD equation, a partial cancellation at $k=0$ occurs between the horizon condition and the ghost self-energy. This cancellation occurs for the always multiplicative renormalizable part coming from $\lambda_{\mu\mu}(0)$.

$$\begin{aligned} 0 =&(N^2-1)D - \langle h(0) \rangle \\ =&(N^2-1)\left\{ D[1+u(0)] + w(0) - G(0)[u(0)+w(0)]^2 \right\}, \end{aligned} \tag{12}$$

This horizon condition is not multiplicatively renormalizable!

$$\begin{aligned} 0 =&Z_C[(N^2-1)D - \langle h(0) \rangle] \\ =&(N^2-1)DZ_C + \lim_{k\to 0} Z_C \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k \\ =&(N^2-1)\left\{ DZ_C[1+u(0)] + Z_C w(0) - Z_C G(0)[u(0)+w(0)]^2 \right\} \\ \neq&(N^2-1)\left\{ D[1+u_R(0)] + w_R(0) - G_R(0)[u_R(0)+w_R(0)]^2 \right\}?, \end{aligned} \tag{13}$$

Note that the renormalization prescription

$$1+u(0) = Z_C^{-1}[1+u_R(0)], \quad w(0) = Z_C^{-1} w_R(0), \quad G_R^{-1}(0) = Z_C^{-1} G^{-1}(0), \tag{14}$$

can not maintain the first horizon condition.

An unconventional method: We recall the SD equation with the horizon condition:

$$G_\Lambda^{-1}(k^2) = \frac{G_\Lambda^{-1}(0) - 2 + G_\Lambda(0)}{D} - \frac{w_\Lambda(0)}{D} - u_\Lambda(0) + u_\Lambda(k^2) + w_\Lambda(k^2). \tag{15}$$

This is unrenormalized version. The UV cutoff $\Lambda$ must be introduced to make the self-energy part $u(k^2)$ finite. [$w(k^2)$ is finite from the begining by some reason. See the next slide] So, $G$ depends on $\Lambda$. $G(k^2, \Lambda)$.

A novel situation occurs by introducing the horizon condition. By the resulting subtraction $u(k^2) - u(0)$, the ultraviolet divergence cancels. Then this is regarded as a self-consistent equation to give a finite ghost function $G(k^2) = \lim_{\Lambda\to\infty} G(k^2, \Lambda) < \infty$.

In other words, the SD equation is self-organized (in a non-perturbative way) to give a finite result for any $k$. $\Longrightarrow$ no need for UV renormalization!

We have the contribution from the remainig term $\Delta_{\mu\mu}(0)$ which is non-zero. Therefore, we have the decoupling solution, $G^{-1}(0) \neq 0$.

$$\lambda_{\mu\nu}^{AB}(k) := \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{m1PI}} = \left[ g_{\mu\nu} u(k^2) + \frac{k_\mu k_\nu}{k^2} w(k^2) \right] \delta^{AB}, \quad (16)$$

The ultraviolet divergence appears in $u$ and not in $w$. So the ultraviolet divergence is proportional to $g_{\mu\nu}$.

$$u(k^2) = \frac{1}{(D-1)(N^2-1)} \left[ g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right] \lambda_{\mu\nu}^{AA}(k), \quad (17a)$$

$$w(k^2) = \frac{-1}{(D-1)(N^2-1)} \left[ g^{\mu\nu} - D\frac{k^\mu k^\nu}{k^2} \right] \lambda_{\mu\nu}^{AA}(k). \quad (17b)$$

The Brown-Pennington projector eliminates the term proportional to $g_{\mu\nu}$.

[power counting]

$\mathrm{dim.}[\mathscr{A}] = (D-2)/2 = \mathrm{dim.}[\mathscr{C}] = \mathrm{dim.}[\bar{\mathscr{C}}]$ means $\mathrm{dim.}[\mathscr{A}\mathscr{C}\mathscr{A}\bar{\mathscr{C}}] = 2D-4$. $\mathrm{dim.}[g] = (4-D)/2$. Therefore, $\mathrm{dim.}[\langle \mathscr{A}\mathscr{C}\mathscr{A}\bar{\mathscr{C}} \rangle_k] = 2D-4-D = D-4$. Thus, for $D=4$, $\lambda_{\mu\nu}^{AB}(k)$ is logarithmic divergent. Only $u$ has divergence, while $w$ is ultraviolet finite. The 1PI part has no massless pole.

For the second horizon condition, a similar situation does occur.

$$G_\Lambda^{-1}(k^2) = \frac{1}{D}[1 + u_\Lambda(0)] - u_\Lambda(0) + u_\Lambda(k^2) + w_\Lambda(k^2), \tag{18}$$

which is also rewritten as

$$G_\Lambda^{-1}(k^2) = \frac{G_\Lambda^{-1}(0)}{D} - \frac{w_\Lambda(0)}{D} - u_\Lambda(0) + u_\Lambda(k^2) + w_\Lambda(k^2). \tag{19}$$

As already shown, due to the linearity, the UV renormalization of this SD equation can be performed within the multiplicative renormalization framework: $1 + u_\Lambda(0) = Z_C^{-1}[1 + u_R(0)]$, $w_\Lambda(0) = Z_C^{-1} w_R(0)$ and $G_\Lambda^{-1}(0) = Z_C^{-1} G_R^{-1}(0)$. Moreover, we can use the same argument as the above: The UV divergence cancels for $u(k^2)$ due to the subtraction $-u_\Lambda(0) + u_\Lambda(k^2)$, while $w_\Lambda(k^2)$ is finite. Therefore, the UV cutoff in $G_\Lambda(k^2)$ is sent to infinity without divergences.

• The renormalization point dependence: From

$$G_R(k^2, \mu^2) = Z_C^{-1}(\mu^2, \Lambda^2) G(k^2, \Lambda^2), \tag{20}$$

we have

$$\frac{G_R(k^2, \mu^2)}{G_R(\mu^2, \mu^2)} = \frac{G(k^2, \Lambda^2)}{G(\mu^2, \Lambda^2)} \tag{21}$$

In particular, at $k^2 = 0$

$$G_R(0, \mu^2) = G_R(\mu^2, \mu^2) \frac{G(0, \Lambda^2)}{G(\mu^2, \Lambda^2)} = G_R(\mu^2, \mu^2) \frac{1 + \sqrt{D}}{G(\mu^2, \Lambda^2)} \tag{22}$$

For the decoupling solution, if we take the renormalization condition, e.g.,

$$G_R(\mu^2, \mu^2) = 1 \Longrightarrow G_R(0, \mu^2) = \frac{G(0)}{G(\mu^2)} = \frac{1 + \sqrt{D}}{G(\mu^2)} \tag{23}$$

For instance, if one chooses $\mu = 1.5$GeV and $D = 4$, then $G(\mu^2) = 1.2$ for a given b.c. $G(0) = 3$. So, we can reproduce the Orsay data: $G_R(0, \mu^2) = 3/1.2 = 2.5$.

[C.S. Fischer, A. Maas and J.M. Pawlowski, arXiv:0810.1987 [hep-ph], Ann. of Phys]
It is only a matter of infrared boundary conditions $G(0)$ whether scaling or decoupling occurs.

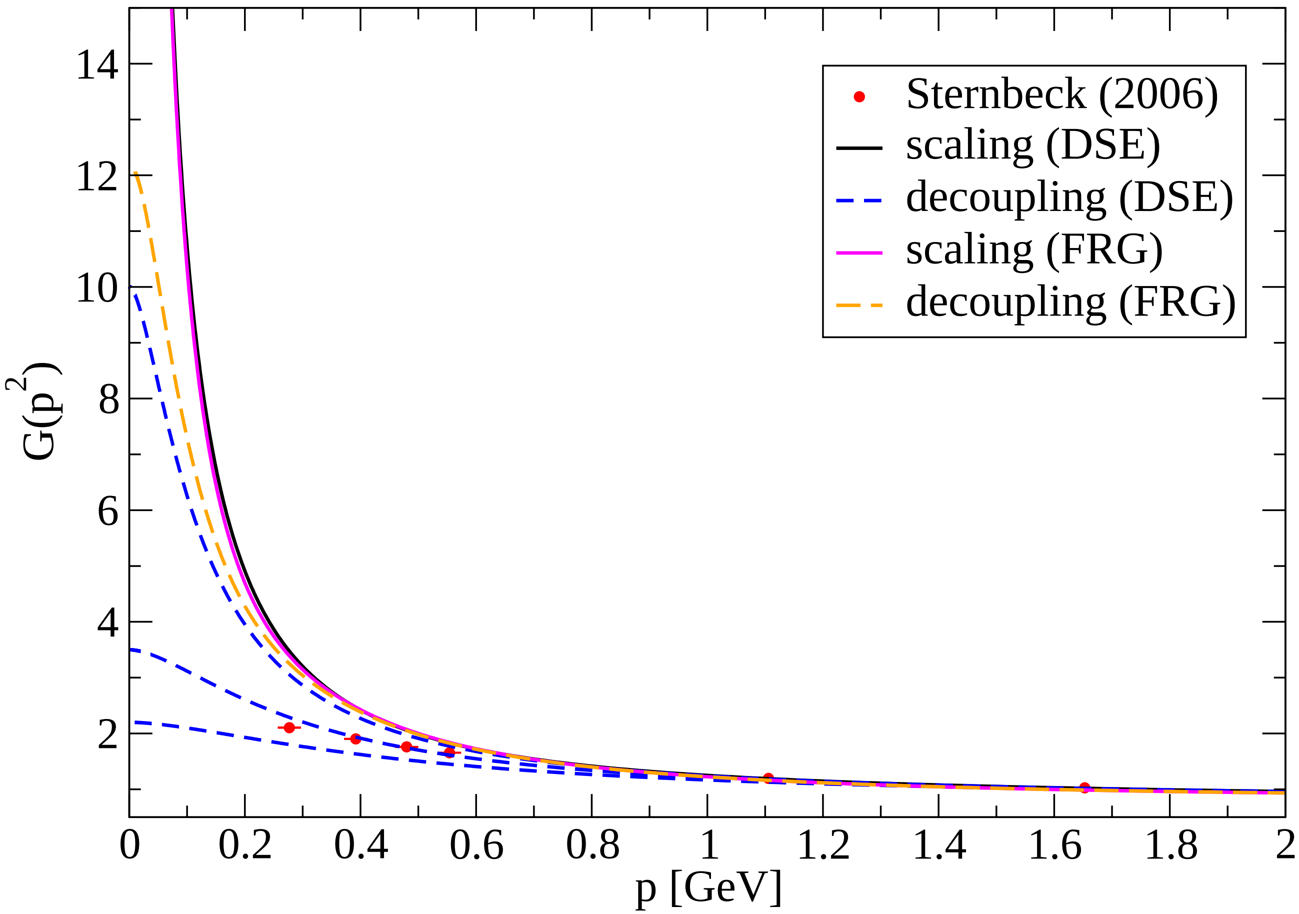



Figure 16: Ghost dressing functions obtained from the Schwinger-Dyson equation and functional renormalization group compared to lattice results in minimal Landau gauge from [Sternbeck, 2006][Bowman et al, 2004].

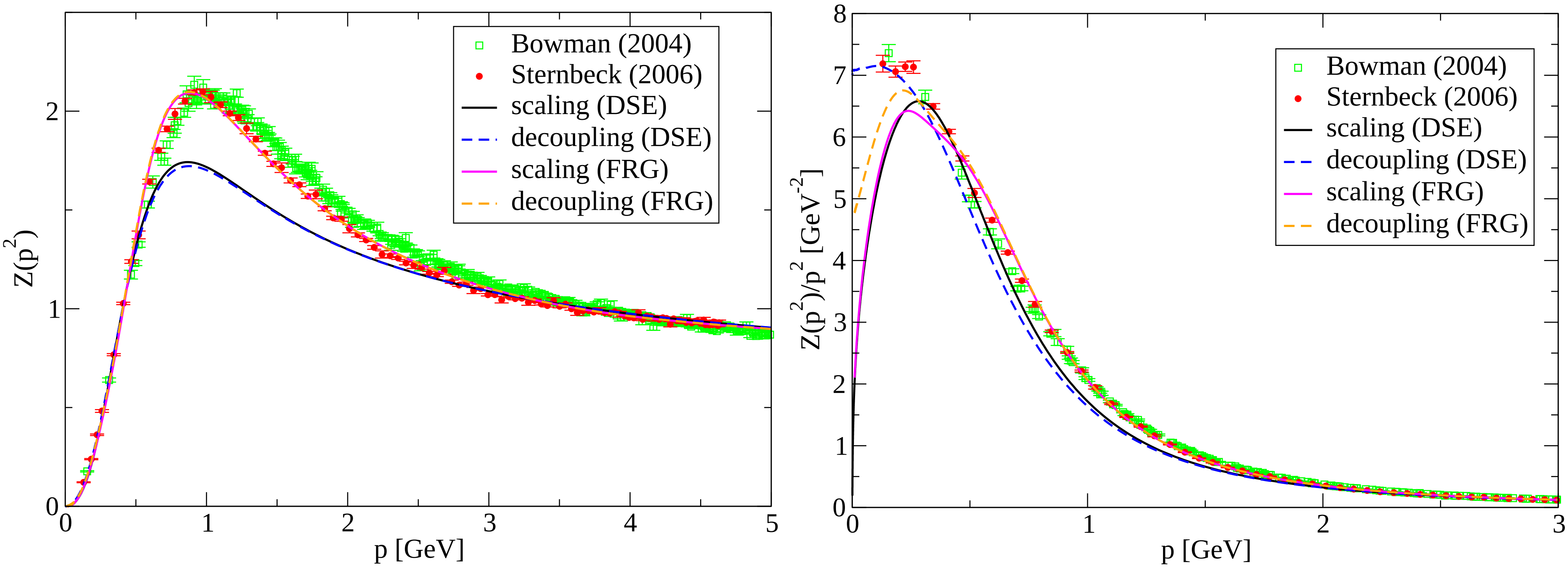



Figure 17: Gluon dressing functions and gluon propagators obtained from the Schwinger-Dyson equation and functional renormalization group compared to lattice results in minimal Landau gauge from [sternbeck06][Bowman:2004jm]. [C.S. Fischer, A. Maas and J.M. Pawlowski, arXiv:0810.1987 [hep-ph]]

# § Conclusion and discussion

We have discussed how the existence of the Gribov horizon modifies the deep infrared behavior of the Landau gauge SU(N) Yang-Mills theory, in the Gribov-Zwanziger framework with the horizon condition $\langle h(x) \rangle = (\dim G) D$.

We examined two horizon functions.

⊙ **For** $h(x) = \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y)$,**[Zwanziger,1989]**
the GZ theory is not multiplicatively renormalizable. However, the SD equation can be UV finite, once the horizon condition is incorporated into the SD equation.
The decoupling solution, $G(0) < \infty$ is allowed to exist, unless $w(0) = -\infty$.
The KO criterion $u(0) = -1$ is not necessarily satisfied except for a special choice of $w(0) = 1/2$ for any $D$, leading to $G(0) = 2$.
For $w(0) = 0$, the boundary values are $G(0) = 1 + \sqrt{D}$ and $u(0) = (-D + \sqrt{D})/(D-1)$ up to renormalization point dependence.

⊙ **For** $h(x) = \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) D_\mu[\mathscr{A}]^{AE}(y)$,**[Zwanziger,1993]**
the GZ theory is multiplicatively renormalizable.
The KO criterion $u(0) = -1$ is satisfied in the unrenormalized and the renormalized cases.
For $w_R(0) = 0$, the scaling solution, $G_R(0) = w_R(0)^{-1} = \infty$, after the renormalization.
For $w_R(0) \neq 0$, the decoupling solution against the claim in the previous literatures.

## ⊙ Schwinger-Dyson equation

We have shown how to incorporate the horizon condition into the Schwinger-Dyson equation for the ghost propagator to discriminate between scaling and decoupling.

$$G^{-1}(k^2) = \frac{\langle h(0) \rangle}{D(N^2 - 1)} + u(k^2) + w(k^2). \tag{1}$$

## ⊙ Renormalization

A possible renormalization scheme and the renormalization point dependence of the decoupling solution has been discussed. This should be compared with [A.C. Aguilar, D. Binosi and J. Papavassiliou, arXiv:0907.0153 [hep-ph]. ]

## ⊙ BRST symmetry and color confinement

How the existence of the horizon is relevant for color confinement. In the Gribov-Zwanziger theory (restricted to the 1st Gribov region), the BRST symmetry is broken by the existence of the horizon. $\boldsymbol{\delta} S_{\rm GZ} = \boldsymbol{\delta} \tilde{S}_\gamma \neq 0$

Nevertheless, there exists a "BRST" like symmetry (without nilpotency [Sorella,0905.1010[hep-th]] or with nilpotency [K.-I. K., 0905.1899[hep-th]]) which leaves the Gribov-Zwanziger action invariant. Then we could apply the Kugo-Ojima idea to the Gribov-Zwanziger theory, which opens the path to searching for the modified color confinement criterion *a la* Kugo and Ojima.

## ⊙ Coulomb gauge

In the Coulomb gauge, there exists the same problem as in the Landau gauge. Be careful!

## ⊙ Quark confinement

[J. Braun, H. Gies and J.M. Pawlowski, Quark Confinement from Color Confinement. e-Print: arXiv:0708.2413 [hep-th]] It is shown that all solutions (decoupling as well as scaling) lead to quark confinement by proving the vanishing of the order parameter of quark confinement, the Polyakov loop.

# Thank you for your attention!

# § A modified BRST transformation

Our main motivation is to find out a modified BRST transformation $\boldsymbol{\delta}'$ such that $\boldsymbol{\delta}'$ leaves the action $S_{\rm YM}^{\rm tot}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}]$ invariant, i.e.,

$$\boldsymbol{\delta}'(S_{\rm YM}^{\rm tot}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}])=0, \tag{1}$$

and $\boldsymbol{\delta}'$ obeys the nilpotency, i.e.,

$$\boldsymbol{\delta}'\boldsymbol{\delta}'=0. \tag{2}$$

Such a transformation could be non-local.
Suppose that $\tilde{S}_\gamma$ is written in the BRST-exact form

$$\begin{aligned}
&S_{\rm GF+FP}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}] \\
=&\int d^Dx\{\mathscr{B}^A\partial_\mu\mathscr{A}_\mu^A - i\bar{\mathscr{C}}^A K^{AB}\mathscr{C}^B + \bar{\xi}_\mu^{CA}K^{AB}\xi_\mu^{CB} - \bar{\omega}_\mu^{CA}K^{AB}\omega_\mu^{CB} \\
&\qquad + i\gamma^{1/2}gf^{ABC}\mathscr{A}_\mu^B\xi_\mu^{AC} + i\gamma^{1/2}gf^{ABC}\mathscr{A}_\mu^B\bar{\xi}_\mu^{AC}\} \qquad (3) \\
=&\int d^Dx\{-i\boldsymbol{\delta}'\left[\bar{\mathscr{C}}^A(\partial_\mu\mathscr{A}_\mu^A)\right] + \boldsymbol{\delta}'[\bar{\omega}_\mu^{CA}(-\partial_\rho D_\rho^{AB}[\mathscr{A}])\xi_\mu^{CB}]\}, \qquad (4)
\end{aligned}$$

The BRST invariance of $S_{\rm GF+FP}+\tilde{S}_\gamma$, i.e., $\boldsymbol{\delta}'(S_{\rm GF+FP}+\tilde{S}_\gamma)=0$, is guaranteed by the nilpotency ($\boldsymbol{\delta}'\boldsymbol{\delta}'=0$) of the modified BRST transformation.

A modified BRST transformation

$$
\begin{aligned}
\boldsymbol{\delta}' \mathscr{A}_\mu^A(x) =& (D_\mu[\mathscr{A}]\mathscr{C}(x))^A, \\
\boldsymbol{\delta}' \mathscr{C}^A(x) =& -\frac{g}{2}(\mathscr{C}(x) \times \mathscr{C}(x))^A, \\
\boldsymbol{\delta}' \bar{\mathscr{C}}^A(x) =& i\mathscr{B}^A(x) + F^A(x), \\
\boldsymbol{\delta}' \mathscr{B}^A(x) =& i\boldsymbol{\delta}' F^A(x), \\
\boldsymbol{\delta}' \xi_\mu^{AB}(x) =& \omega_\mu^{AB}(x) + G_\mu^{AB}(x), \\
\boldsymbol{\delta}' \omega_\mu^{AB}(x) =& -\boldsymbol{\delta}' G_\mu^{AB}(x), \\
\boldsymbol{\delta}' \bar{\omega}_\mu^{AB}(x) =& \bar{\xi}_\mu^{AB}(x) + H_\mu^{AB}(x), \\
\boldsymbol{\delta}' \bar{\xi}_\mu^{AB}(x) =& -\boldsymbol{\delta}' H_\mu^{AB}(x),
\end{aligned}
\tag{5}
$$

where

$$F^A(x) = \gamma^{1/2} \int d^D y \Delta^{-1}(x, y) g f^{ABC} \partial_\mu \bar{\xi}_\mu^{BC}(y), \tag{6}$$

$$G_\mu^{AB}(x) = \int d^D y (K^{-1})^{AC}(x, y) \partial_\rho [g f^{CFE} (D_\rho \mathscr{C})^F(y) \xi_\mu^{EB}(y)], \tag{7}$$

$$H_\mu^{AB}(x) = \int d^D y i\gamma^{1/2} g f^{BCE} \mathscr{A}_\mu^C(y) (K^{-1})^{EA}(y, x). \tag{8}$$

Note that $G$ does not vanish even in the limit $\gamma \to 0$ and the modified BRST transformation $\boldsymbol{\delta}'\xi_\mu^{AB}(x)$ has the part $G$ involving the Yang-Mills field and the ghost field. Even in the limit, therefore, the horizon term is not decoupled from the usual Yang-Mills-Faddeev-Popov theory. This issue is cured by redefining the auxiliary field $\omega_\mu^{AB}(x)$, i.e., shifting it by $G_\mu^{AB}(x)$

$$\omega'^{AB}_\mu(x) := \omega_\mu^{AB}(x) + G_\mu^{AB}(x). \tag{9}$$

Then the modified BRST transformation is simplified

$$\begin{aligned}
\boldsymbol{\delta}'\mathscr{A}_\mu^A(x) =&(D_\mu[\mathscr{A}]\mathscr{C}(x))^A, && \text{(10a)}\\
\boldsymbol{\delta}'\mathscr{C}^A(x) =&-\frac{g}{2}(\mathscr{C}(x)\times\mathscr{C}(x))^A, && \text{(10b)}\\
\boldsymbol{\delta}'\bar{\mathscr{C}}^A(x) =&i\mathscr{B}^A(x) + F^A(x), && \text{(10c)}\\
\boldsymbol{\delta}'\mathscr{B}^A(x) =&i\boldsymbol{\delta}'F^A(x), && \text{(10d)}\\
\boldsymbol{\delta}'\xi_\mu^{AB}(x) =&\omega'^{AB}_\mu(x), && \text{(10e)}\\
\boldsymbol{\delta}'\omega'^{AB}_\mu(x) =&0, && \text{(10f)}\\
\boldsymbol{\delta}'\bar{\omega}_\mu^{AB}(x) =&\bar{\xi}_\mu^{AB}(x) + H_\mu^{AB}(x), && \text{(10g)}\\
\boldsymbol{\delta}'\bar{\xi}_\mu^{AB}(x) =&-\boldsymbol{\delta}'H_\mu^{AB}(x), && \text{(10h)}
\end{aligned}$$

This result should be compared with the result of Sorella,0905.1010[hep-th] , another modified BRST transformation without nilpotency.

Since we have found a modified BRST transformation which leaves the Gribov-Zwanziger action invariant, then we could apply the Kugo-Ojima idea to the Gribov-Zwanziger theory, which opens the path to searching for the modified color confinement criterion $a\ la$ Kugo and Ojima. ...

# § On the gluon dressing function

The purpose is to search the solution which is consistent with the general principles of quantized gauge field theory:

- Non-perturbative multiplicative renormalizability
- Analyticity
- Spectral condition
- Poincaré group structure

Then the gluon dressing function vanishes in the IR limit $p^2 \to 0$.

The gluon propagator can be finite and non-zero.

[K.-I. K., hep-th/0303251]

Short version [K.-I. K., hep-lat/0309142]

First, we consider the case of $w(0) = 0$ [supported by numerical calculations].

- a new relationship $\quad \langle h(0) \rangle = (N^2 - 1) \left\{ -Du(0) + \frac{u(0)^2}{1 + u(0)} \right\},$ (1)
- the horizon condition $\quad \langle h(0) \rangle = (N^2 - 1)D,$ (2)

$$\to (D-1)u(0)^2 + 2Du(0) + D = 0, \quad u(0) = (-D \pm \sqrt{D})/(D-1) \tag{3}$$
$$\to G(0)^2 - 2G(0) + 1 - D = 0, \quad G(0) = 1 \pm \sqrt{D} \tag{4}$$

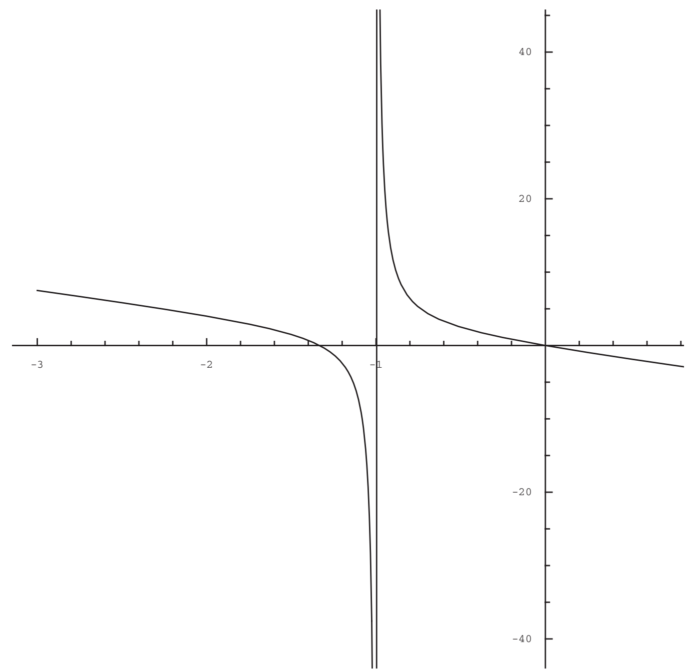



Figure 18: The plot of $\langle h(0) \rangle$ versus $u(0)$ for $D = 4$.

For $D = 4$, $3u(0)^2 + 8u(0) + 4 = 0$ has solutions $u(0) = -2/3, -2$. We obtain irrespective of the number of color N

$$u(0) = -\frac{2}{3}, \quad G(0) = [1 + u(0)]^{-1} = 3. \tag{5}$$

The ghost dressing function $G(k^2)$ is finite even in the deep infrared limit $k^2 \to 0$.

- Second, we consider the (unlikely) case of $w(0) \neq 0$. [See numerical results]

The horizon condition alone is not sufficient to determine both $u(0)$ and $w(0)$. Suppose the Kugo-Ojima confinement criterion is satisfied $u(0) = -1$. Then the horizon condition is

$$\langle h(0) \rangle = -(N^2-1)\left\{ -D + 1 + \frac{-1+w(0)}{w(0)} \right\} \cong (N^2-1)D. \tag{6}$$

This leads to the value of $w(0)$ irrespective of the spacetime dimension $D$ and the number of color $N$:

$$w(0) = 1/2 \quad \text{for any} \quad D. \tag{7}$$

Even if $u(0) = -1$, therefore, the ghost propagator behaves like free $1/k^2$ at $k = 0$, no more singular than $1/k^2$: irrespective of the spacetime dimension $D$ and the number of color $N$

$$\lim_{k^2 \to 0} [-k^2 \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k]^{-1} = \delta^{AB} w(0) = \frac{1}{2}\delta^{AB} \neq 0, \quad G(0) = 2. \tag{8}$$

Thus, the ghost propagator behaves like free at low momenta, while the gluon propagator is non-vanishing at low momenta The original Gribov prediction is wrong?

$$
\begin{aligned}
&\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k \\
:=&\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI} + \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^C \rangle_k^{1PI} \langle \mathscr{C}^C \bar{\mathscr{C}}^D \rangle_k \langle \mathscr{C}^D (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{1P} \\
=&\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI} \\
&+ k_\sigma k_\rho \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\sigma \times \bar{\mathscr{C}})^C \rangle_k^{m1PI} \langle \mathscr{C}^C \bar{\mathscr{C}}^D \rangle_k \langle (g\mathscr{A}_\rho \times \mathscr{C})^D (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI} \\
=&[g_{\mu\nu} u(k^2) + k_\mu k_\nu v(k^2)] \delta^{AB} + k_\mu k_\nu \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k [u(k^2) + k^2 v(k^2)]^2,
\end{aligned} \tag{9}
$$

After performing Lorentz and color contraction, indeed, we obtain the same result:

$$
\begin{aligned}
&\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k^{\alpha=0} \\
=&(N^2-1) \left\{ D u(k^2) + k^2 v(k^2) + k^2 \langle \mathscr{C} \bar{\mathscr{C}} \rangle_k [u(k^2) + k^2 v(k^2)]^2 \right\} \\
=&(N^2-1) \left\{ (D-1)\, u(k^2) + \frac{u(k^2) + k^2 v(k^2)}{1 + u(k^2) + k^2 v(k^2)} \right\}.
\end{aligned} \tag{10}
$$

If we neglect the term,

$$
- \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k^{\alpha=0} = -(N^2-1) \left\{ D u(k^2) + k^2 v(k^2) \right\}. \tag{11}
$$

# § Renormalization?

renormalization of fields

$$\mathscr{A}_0 = Z_3^{1/2}\mathscr{A}_R, \quad \mathscr{C}_0 = \tilde{Z}_3^{1/2}\mathscr{C}_R, \quad \bar{\mathscr{C}_0} = \tilde{Z}_3^{1/2}\bar{\mathscr{C}_R}, \tag{1}$$

$$\psi_0 = Z_2^{-1/2}\psi_R, \quad \bar{\psi}_0 = Z_2^{1/2}\bar{\psi}_R, \quad \ldots \tag{2}$$

renormalization of the coupling constant

$$g_0 = Z_3^{-3/2}Z_1 g_R = Z_3^{-1/2}\tilde{Z}_3^{-1}\tilde{Z}_1 g_R = Z_3^{-1}Z_4^{1/2}g_R = Z_3^{-1/2}Z_2^{-1}Z_1^F g_R = \ldots \tag{3}$$

Slavnov-Taylor identity

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_1}{Z_4} = \frac{Z_2}{Z_1^F} = \ldots \tag{4}$$

In the Landau gauge

$$\tilde{Z}_1 = 1, \tag{5}$$

$$\frac{Z_3}{Z_1} = \tilde{Z}_3 = \frac{1}{1+u(0)+w(0)} \tag{6}$$

In the n.b.d of $p^2 = 0$,

$$Z_1(p^2) = \frac{1}{1+u(0)+w(0)}Z_3(p^2) \cong \frac{c_0}{1+u(0)+w(0)}p^2, \quad Z_3(p^2) \cong c_0 p^2 \tag{7}$$

Note that

$$\mathscr{B}_0 = Z_3^{-1/2}\mathscr{B}_R, \quad (D_\mu \mathscr{C})_0 = \tilde{Z}_3^{-1/2}(D_\mu \mathscr{C})_R, \tag{8}$$

Gribov-Zwanziger theory [Zwanziger, 1992, section 6.3]

The auxiliary field $\xi, \bar{\xi}$ has the same renormalization constant as the ghost, antighost:

$$\xi_0 = \tilde{Z}_3^{1/2}\xi_R, \quad \bar{\xi}_0 = \tilde{Z}_3^{1/2}\bar{\xi}_R, \tag{9}$$

$$\gamma_0^{1/2}\mathscr{A}_0\xi_0 = \gamma_R^{1/2}\mathscr{A}_R\xi_R \to \gamma_0^{1/2} = Z_3^{-1/2}\tilde{Z}_3^{-1/2}\gamma_R^{1/2} \tag{10}$$

horizon condition

$$(N^2-1)D = i\gamma_0^{-1/2}\langle g_0^2\mathscr{A}_0\xi_0\rangle = i\tilde{Z}_3^{-1}\tilde{Z}_1^2\gamma_R^{-1/2}\langle g_R^2\mathscr{A}_R\xi_R\rangle \tag{11}$$

Kugo-Ojima parameter

$$u_0 \sim \langle (D\mathscr{C})_0(g_0\mathscr{A}_0 \times \bar{\mathscr{C}}_0)\rangle = \tilde{Z}_3^{-1}\tilde{Z}_1\langle (D\mathscr{C})_R(g_R\mathscr{A}_R \times \bar{\mathscr{C}}_R)\rangle = \tilde{Z}_3^{-1}\tilde{Z}_1 u_R \tag{12}$$

$$\langle h_0\rangle \sim \langle (g_0\mathscr{A}_0 \times \mathscr{C}_0)(g_0\mathscr{A}_0 \times \bar{\mathscr{C}}_0)\rangle = \tilde{Z}_3^{-1}\tilde{Z}_1^2\langle (g_R\mathscr{A}_R \times \mathscr{C}_R)(g_R\mathscr{A}_R \times \bar{\mathscr{C}}_R)\rangle \tag{13}$$

# § BRST quantization of Yang-Mills theory

• For the $D$-dimensional SU(N) Yang-Mills theory, BRST quantization for the usual Faddeev-Popov approach,

$$Z := \int [d\mathscr{A}][d\mathscr{B}][d\mathscr{C}][d\bar{\mathscr{C}}] \exp\{iS_{YM}^{tot}\}, \tag{1}$$

where

$$\begin{aligned} S_{YM}^{tot} :=& S_{YM} + S_{GF+FP}, \\ S_{YM} :=& -\int d^D x \frac{1}{4} \mathscr{F}_{\mu\nu} \cdot \mathscr{F}_{\mu\nu}, \\ S_{GF+FP} :=& \int d^D x \left\{ \mathscr{B} \cdot \partial_\mu \mathscr{A}_\mu + \frac{\alpha}{2} \mathscr{B} \cdot \mathscr{B} + i\bar{\mathscr{C}} \cdot \partial_\mu D_\mu \mathscr{C} \right\}, \\ \mathscr{F}_{\mu\nu} :=& \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + g\mathscr{A}_\mu \times \mathscr{A}_\nu, \\ D_\mu \mathscr{C} :=& (\partial_\mu + g\mathscr{A}_\mu \times)\mathscr{C}, \end{aligned} \tag{2}$$

and the dot and the cross are defined as

$$\mathscr{A} \cdot \mathscr{B} := \mathscr{A}^A \mathscr{B}^A, \quad (\mathscr{A} \times \mathscr{B})^A := f^{ABC} \mathscr{A}^B \mathscr{B}^C, \tag{3}$$