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Quark confinement: Recent progress based on dual superconductor picture
クォーク閉じ込め：双対超伝導描像に基づく最近の発展

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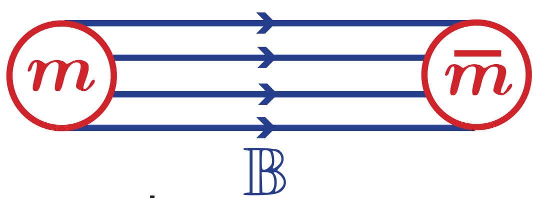
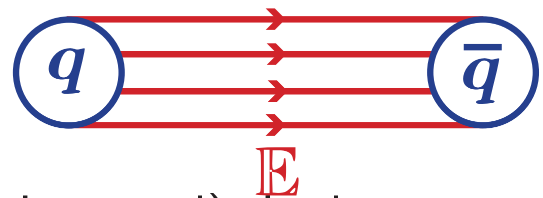
Chapter:
Introduction
Magnetic monopoles in gauge field theories

§ Dual superconductor picture for confinement





(Left panel) superconductor (Right panel) dual superconductor

Superconductivity (type II)

condensation of electric charges (Cooper pairs)

$\Downarrow$

Meissner effect: formation of Abrikosov string (magnetic flux tube)

connecting a monopole m and an anti-monopole $\bar{m}$

$\Downarrow$

linear potential between a monopole m and an anti-monopole $\bar{m}$

$\Longleftrightarrow$ electric-magnetic dual [Nambu, 1974] ['tHooft, 1975][Mandelstam, 1976]

linear confining potential between q and $\bar{q}$

$\Uparrow$

dual Meissner effect: formation of a hadron string

(electric flux tube) connecting q and $\bar{q}$

$\Uparrow$

condensation of **magnetic monopoles**

· Dual superconductivity in YM theory !?

§ Magnetic monopoles in gauge field theories

1) In Electro-magetism,

Dirac magnetic monopole

2) In non-Abelian gauge theory with (adjoint) matter fields, e.g., Georgi-Glashow model,

't Hooft-Polyakov magnetic monopole

3) How can the magnetic monopole be defined in pure non-Abelian gauge theory (in absence of matter fields)?
The magnetic monopole is a basic ingredient in dual superconductivity picture [Nambu 1974, Mandelstam 1976, 't Hooft 1978] for understanding quark confinement in QCD.

1. 't Hooft (Abelian projection, partial gauge fixing)
   [Nucl. Phys. B190, 455 (1981)]

2. Cho & Faddeev-Niemi (field decomposition, new variables)
   [Phys. Rev. D21, 1080 (1980)] [Phys. Rev. Lett. 82, 1624 (1999)]...

The purpose of this talk is to give a short review of recent developments on the second method from the viewpoint of quark confinement.
In particular, I emphasize some aspects of the second method superior to the first one.

§ 't Hooft Abelian projection and magnetic monopole

Consider the (pure) Yang-Mills theory with the gauge group $G = SU(N)$ on $\mathbb{R}^D$.

(1) Let $\chi(x)$ be a Lie-algebra $\mathscr{G}$-valued functional of the Yang-Mills field $\mathscr{A}_\mu(x)$. Suppose that it transforms in the adjoint representation under the gauge transformation:

$$\chi(x) \to \chi'(x) := U(x)\chi(x)U^\dagger(x) \in \mathscr{G} = su(N), \quad U(x) \in G, \quad x \in \mathbb{R}^D. \tag{1}$$

(2) Diagonalize the Hermitian $\chi(x)$ by choosing a suitable unitary matrix $U(x) \in G$

$$\chi'(x) = \mathrm{diag}(\lambda_1(x), \lambda_2(x), \cdots, \lambda_N(x)). \tag{2}$$

This is regarded as a partial gauge fixing, if $\chi(x)$ is a gauge-dependent quantity.

(2a) At non-degenerate points $x \in \mathbb{R}^D$ of spacetime, the gauge group G is partially fixed, leaving a subgroup H unfixed, i.e., a partial gauge fixing:

$$G = SU(N) \to H = U(1)^{N-1} \times \mathrm{Weyl}. \tag{3}$$

(2b) At degenerate points $x_0 \in \mathbb{R}^D$, $\lambda_j(x_0) = \lambda_k(x_0)$ $(j \neq k = 1, \cdots, N)$, a magnetic monopole appears in the diagonal part of $\mathscr{A}_\mu(x)$ (gauge fixing defects).

$G = SU(N)$ non-Abelian Yang-Mills field
$\to H = U(1)^{N-1}$ Abelian gauge field + magnetic monopoles + electrically charged matter field ['t Hooft, 1981] e.g., $\chi(x) = \mathscr{F}_{12}(x)$, $\mathscr{F}_{\mu\nu}^2$, $\mathscr{F}_{\mu\nu}(x) D^2 \mathscr{F}_{\mu\nu}(x)$

For the SU(2) matrix $U(x) = e^{-i\gamma(x)\sigma_3(x)/2} e^{-i\beta(x)\sigma_2(x)/2} e^{-i\alpha(x)\sigma_3(x)/2}$ diagonalizing the Hermitian $\chi(x)$, the diagonal part of the gauge transformed Yang-Mills field

$$ig^{-1} U(x) \partial_\mu U^\dagger(x) = g^{-1} \frac{1}{2} \begin{pmatrix} \cos\beta \partial_\mu \alpha + \partial_\mu \gamma & [-i\partial_\mu \beta - \sin\beta \partial_\mu \gamma] e^{i\alpha} \\ [i\partial_\mu \beta - \sin\beta \partial_\mu \gamma] e^{i\alpha} & -[\cos\beta \partial_\mu \alpha + \partial_\mu \gamma] \end{pmatrix} = \mathscr{V}_\mu^A \sigma_A / 2 \tag{4}$$

contains the singular potential of the Dirac type.

$$\mathscr{V}_\mu^3 = g^{-1} [\cos\beta \partial_\mu \alpha + \partial_\mu \gamma]. \tag{5}$$

The $D = 3$ case agrees with the Dirac magnetic potential by choosing $\alpha = \varphi$, $\beta = \theta$, $\gamma = \gamma(\varphi)$ (expressing the degenerate point)

$$\mathscr{V}_\mu^3 = \frac{g^{-1}}{r} \frac{\cos\theta + \partial_\varphi \gamma}{\sin\theta} \mathbf{e}_\varphi \tag{6}$$

Remarkable achievements in Abelian projection (Maximal Abelian gauge)

⊙ Abelian dominance in the string tension

- T. Suzuki and I. Yotsuyanagi, Phys.Rev.D42:4257-4260,1990.

⊙ Magentic monopole dominance in the string tension

- J.D.Stack, S.D.Neiman, R.J.Wensley, Phys.Rev.D50:3399-3405,1994. hep-lat/9404014
- H. Shiba and T. Suzuki, Phys.Lett.B333:461-466,1994. hep-lat/9404015

⊙ Gribov copy effects

- G.S. Bali, V. Bornyakov, M. Muller-Preussker and K. Schilling, Phys.Rev.D54:2863-2875,1996. hep-lat/9603012

⊙ Off-diagonal gluon mass generation

- K. Amemiya and H. Suganuma, Phys.Rev.D60:114509,1999. hep-lat/9811035

⊙ Asymptotic freedom in an effective theory of dual Ginzburg-Landau type

- M. Quandt and H. Reinhardt, Int.J.Mod.Phys.A13:4049-4076,1998. hep-th/9707185
- K.-I. Kondo, Phys.Rev.D57:7467-7487,1998. hep-th/9709109

⊙ Hidden SUSY in a renormalizable MAG and dimensional reduction

- K.-I. Kondo, Phys.Rev.D58:105019,1998. hep-th/9801024
- K.-I. Kondo, Phys.Rev.D58:105016,1998. hep-th/9805153 ⋮

§ Maximal Abelian gauge (MAG) and magnetic monopoles

• quark confinement follows from the area law of the Wilson loop average [Wilson,1974]

$$\text{Non-Abelian Wilson loop} \quad \left\langle \mathrm{tr}\left[\mathscr{P} \exp\left\{ ig \oint_C dx^\mu \mathscr{A}_\mu(x) \right\} \right] \right\rangle_{\mathrm{YM}}^{\text{no GF}} \sim e^{-\sigma_{NA}|S|}, \quad (1)$$

• Numerical simulation on the lattice after imposing the Maximal Abelian gauge (MAG): for the SU(2) Cartan decomposition: $\mathscr{A}_\mu = A_\mu^a \frac{\sigma^a}{2} + A_\mu^3 \frac{\sigma^3}{2}$ $(a=1,2)$, $\mathscr{A}_\mu \to A_\mu^3 \frac{\sigma^3}{2}$

$$\text{Abelian-projected Wilson loop} \quad \left\langle \exp\left\{ ig \oint_C dx^\mu A_\mu^3(x) \right\} \right\rangle_{\mathrm{YM}}^{\mathrm{MAG}} \sim e^{-\sigma_{Abel}|S|} \quad !? \quad (2)$$

The continuum form of MAG is $[\partial_\mu \delta^{ab} - g\epsilon^{ab3} A_\mu^3(x)] A_\mu^b(x) = 0$ $(a,b=1,2)$.

· Abelian dominance $\Leftrightarrow$ $\sigma_{Abel} \sim \sigma_{NA}$ (92±4)% [Suzuki & Yotsuyanagi,PRD42,4257,1990]

$$A_\mu^3 = \text{Monopole part} + \text{Photon part}, \quad (3)$$

· Monopole dominance $\Leftrightarrow$ $\sigma_{monopole} \sim \sigma_{Abel}$ (95)%
[Stack, Neiman and Wensley, hep-lat/9404014][Shiba & Suzuki, hep-lat/9404015]

Maximal Abelian gauge $\equiv$ a partial gauge fixing $G = SU(N) \to H = U(1)^{N-1}$: the gauge freedom $\mathscr{A}_\mu(x) \to \mathscr{A}_\mu^\Omega(x) := \Omega(x)[\mathscr{A}_\mu(x) + ig^{-1}\partial_\mu]\Omega^{-1}(x)$ is used to transform the gauge variable as close as possible to the Abelian components for the maximal torus subgroup H of the gauge group G.

The magnetic monopole of the Dirac type appears in the diagonal part A_μ^3 of $\mathscr{A}_\mu(x)$ as defects of gauge fixing procedure.

MAG is given by minimizing the function $F_{\rm MAG}$ w.r.t. the gauge transformation Ω.

$$\min_\Omega F_{\rm MAG}[\mathscr{A}^\Omega], \quad F_{\rm MAG}[\mathscr{A}] := \frac{1}{2}(A_\mu^a, A_\mu^a) = \int d^Dx \frac{1}{2} A_\mu^a(x) A_\mu^a(x) \quad (a = 1, 2) \tag{4}$$

$$\delta_\omega F_{\rm MAG} = (\delta_\omega A_\mu^a, A_\mu^a) = ((D_\mu[A]\omega)^a, A_\mu^a) = -(\omega^a, D_\mu^{ab}[A^3]A_\mu^b) \tag{5}$$

The residual U(1) exists.
cf. Lorentz gauge (Landau gauge) $G = SU(N) \to H = \{0\}$:

$$\min_\Omega F_{\rm L}[\mathscr{A}^\Omega], \quad F_L[\mathscr{A}] := \frac{1}{2}(\mathscr{A}_\mu^A, \mathscr{A}_\mu^A) = \int d^Dx \frac{1}{2}\mathscr{A}_\mu^A(x)\mathscr{A}_\mu^A(x) \quad (A = 1, 2, 3) \tag{6}$$

$$\delta_\omega F_L = (\delta_\omega \mathscr{A}_\mu^A, \mathscr{A}_\mu^A) = ((D_\mu[\mathscr{A}]\omega)^A, \mathscr{A}_\mu^A) = -(\omega^A, (D_\mu[\mathscr{A}]\mathscr{A}_\mu)^A) = -(\omega^A, \partial_\mu \mathscr{A}_\mu^A)$$

$$\delta_\omega^2 F_L = -(\omega^A, \partial_\mu \delta_\omega \mathscr{A}_\mu^A) = (\omega^A, (-\partial_\mu D_\mu[\mathscr{A}])^{AB}\omega^B) \quad \text{FP operator}$$

⊙ Problems:

• The naive Abelian projection and the MAG break color symmetry explicitly.

• Abelian dominance in the string tension ... has never been observed in gauge fixings other than MAG.

The criticism: The magentic monopole and the resulting dual superconductivity in Yang-Mills theory might be a gauge artifact?

In order to establish the gauge-invariant dual superconductivity in Yang-Mills theory, we must solve the questions:

1. How to extract the "Abelian" part responsible for dual superconductivity from the non-Abelian gauge theory in the gauge-independent way (without losing characteristic features of non-Abelian gauge theory, e.g., asymptotic freedom).

2. How to define the magnetic monopole to be condensed in Yang-Mills theory even in absence of any matter field in the gauge-invariant way (cf. Georgi-Glashow model).

The second method *a la* Cho-Faddev-Niemi sweeps away all the criticism.

§ Cho-Faddeev-Niemi decomposition

Question: If we find the decomposition of the SU(2) gauge field $\mathbb{A}_\mu(x) = \mathbb{A}_\mu^A(x)\sigma^A/2$,

$$\mathbb{A}_\mu(x) = \mathbb{V}_\mu(x) + \mathbb{X}_\mu(x),$$

such that the field strength $\mathbb{F}_{\mu\nu}[\mathbb{V}]$ is proportional to the unit field $\mathbf{n}$ (i.e., $\mathbf{n}\cdot\mathbf{n} = 1$):

$$\mathbb{F}_{\mu\nu}[\mathbb{V}](x) := \partial_\mu \mathbb{V}_\nu(x) - \partial_\nu \mathbb{V}_\mu(x) + g\mathbb{V}_\mu(x) \times \mathbb{V}_\nu(x) = f_{\mu\nu}(x)\mathbf{n}(x)$$

and that $\mathbb{F}_{\mu\nu}[\mathbb{V}]$ and $\mathbf{n}$ transform in the adjoint rep. under the gauge transformation:

$$\mathbb{F}_{\mu\nu}[\mathbb{V}](x) \to U(x)\mathbb{F}_{\mu\nu}[\mathbb{V}](x)U^\dagger(x), \quad \mathbf{n}(x) \to U(x)\mathbf{n}(x)U^\dagger(x),$$

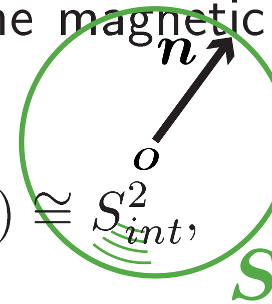
Then we can introduce a gauge-invariant magnetic monopole current by

$$k_\mu(x) = \partial_\nu {}^*f_{\mu\nu}(x) = (1/2)\epsilon_{\mu\nu\rho\sigma}\partial_\nu f^{\rho\sigma}(x),$$

since $f_{\mu\nu}$ is gauge invariant:

$$f_{\mu\nu} := \mathbf{n}\cdot\mathbb{F}_{\mu\nu}[\mathbb{V}] \to f_{\mu\nu}$$

• Magnetic charge quantization: The non-vanishing magnetic charge is obtained without introducing Dirac singularities in c_μ. Even in the classical level, the magnetic charge obeys the quantization condition of Dirac type.



$$\boldsymbol{n}(x) := \begin{pmatrix} n^1(x) \\ n^2(x) \\ n^3(x) \end{pmatrix} = \begin{pmatrix} \sin\beta(x)\cos\alpha(x) \\ \sin\beta(x)\sin\alpha(x) \\ \cos\beta(x) \end{pmatrix} \in SU(2)/U(1) \cong S^2_{int},$$

The magnetic charge g_m is nothing but a number of times S^2_{int} is wrapped by a mapping from S^2_{phys} to S^2_{int}. $[\Pi_2(SU(2)/U(1)) = \Pi_2(S^2) = \mathbb{Z}]$

$$\begin{aligned} g_m &:= \int d^3x k_0 = \int d^3x \partial_i \left(\frac{1}{2}\epsilon^{ijk} f_{jk}\right) \\ &= \oint_{S^2_{phy}} d\sigma_{jk} g^{-1} \boldsymbol{n} \cdot (\partial_j \boldsymbol{n} \times \partial_k \boldsymbol{n}) = g^{-1} \oint_{S^2_{phy}} d\sigma_{jk} \sin\beta \frac{\partial(\beta,\alpha)}{\partial(x^j,x^k)} \\ &= g^{-1} \oint_{S^2_{int}} \sin\beta d\beta d\alpha = 4\pi g^{-1} n \quad (n = 0, \pm 1, \cdots) \end{aligned}$$

where $\frac{\partial(\beta,\alpha)}{\partial(x^\mu,x^\nu)}$ is the Jacobian: $(x^\mu, x^\nu) \in S^2_{phy} \to (\beta,\alpha) \in S^2_{int} \simeq SU(2)/U(1)$ and S^2_{int} is a surface of a unit sphere with area 4π.

Is such a decomposition (spin-charge separation) possible?

Yes!: The answer to this question was given by Cho (1980) [Duan and De (1979)] as

$$\begin{aligned}\mathbb{A}_\mu(x) =& \mathbb{V}_\mu(x) + \mathbb{X}_\mu(x),\\ &\mathbb{V}_\mu(x) = c_\mu(x)\mathbf{n}(x) + g^{-1}\partial_\mu \mathbf{n}(x) \times \mathbf{n}(x) (\leftarrow \text{Cho connection})\\ &\qquad c_\mu(x) = \mathbb{A}_\mu(x) \cdot \mathbf{n}(x),\\ &\mathbb{X}_\mu(x) = g^{-1}\mathbf{n}(x) \times D_\mu[\mathbb{A}]\mathbf{n}(x) \quad (D_\mu[\mathbb{A}] := \partial_\mu + g\mathbb{A}_\mu \times)\end{aligned}$$

The field strength $\mathbb{F}_{\mu\nu}[\mathbb{V}]$ is found to be proportional to $\mathbf{n}$:

$$\mathbb{F}_{\mu\nu}[\mathbb{V}] := \partial_\mu \mathbb{V}_\nu - \partial_\nu \mathbb{V}_\mu + g\mathbb{V}_\mu \times \mathbb{V}_\nu = \mathbf{n}[\partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})]$$

Then we have a gauge-invariant field strength:

$$f_{\mu\nu} := \mathbf{n} \cdot \mathbb{F}_{\mu\nu}[\mathbb{V}] = \partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})$$

Note: Remember this is the same form as the 'tHooft-Polyakov tensor for the magnetic monopole, if the color unit field is the normalized adjoint scalar field in the Georgi–Glashow model: $\mathbf{n}^A(x) \leftrightarrow \hat{\phi}^A(x) := \phi^A(x)/||\phi(x)||$.

The role of the color field $\boldsymbol{n}(x) \in G/\tilde{H}$:

• The color field $\boldsymbol{n}(x)$ carries topological defects without introducing singularities in the gauge potential, e.g., magnetic monopole, knot soliton, ...

• The color field $\boldsymbol{n}(x)$ recovers color symmetry which will be lost in the conventional Abelian projection, the MA gauge.

$$
\begin{aligned}
\mathbf{n}(x) = (0,0,1) \Longrightarrow \mathbb{A}_\mu(x) =& \mathbb{V}_\mu(x) + \mathbb{X}_\mu(x), \\
& \mathbb{V}_\mu(x) = (0,0,c_\mu(x)), \qquad c_\mu(x) = \mathbb{A}_\mu^3(x), \\
& \mathbb{X}_\mu(x) = (\mathbb{A}_\mu^1(x), \mathbb{A}_\mu^2(x), 0)
\end{aligned}
$$

Suppose that $\mathbf{n}(x)$ is given as a functional of $\mathbb{A}_\mu(x)$, i.e., $\mathbf{n}(x) = \mathbf{n}_{\mathscr{A}}(x)$. Then, by solving two defining equations:

(i) covariant constantness (integrability) of color field $\mathbf{n}$ in $\mathbb{V}_\mu$: $\quad D_\mu[\mathbb{V}]\mathbf{n}(x) = 0$

(ii) orthogonality of $\mathbb{X}_\mu(x)$ to $\mathbf{n}(x)$: $\quad \mathbb{X}_\mu(x) \cdot \mathbf{n}(x) = 0$

$\mathbb{V}_\mu$ and $\mathbb{X}_\mu$ are uniquely determined by $\mathbb{A}_\mu(x)$ and $\mathbf{n}$.

Chapter:
Reformulating Yang-Mills theory based on change of variables

§ Reformulation in terms of new variables

We wish to obtain a new reformulation of Yang-Mills theory:

SU(2) Yang-Mills theory written in terms of $\mathbb{A}_\mu^A(x)$ $(A=1,2,3)$	$\Longleftrightarrow$ change of variables	A reformulated SU(2) Yang-Mills theory written in terms of new variables: $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ $(A=1,2,3)$

The following issues must be fixed for two theories to be equipollent in the quantum level:

1. How $\mathbf{n}(x)$ is determined from $\mathbb{A}_\mu(x)$?

   [This was assumed so far. We must give a procedure to achieve this.]

2. How the mismatch between two set of variables is solved?

   [The new variables have two extra degrees of freedom which should be eliminated by imposing appropriate constraints.]

- Counting the degrees of freedom: D-dim. SU(2) Yang-Mills

before	$\mathscr{A}_\mu^A$: 3D				total 3D
after	n^A: $3-1=2$	X_μ^A: $3D-D=2D$	C_μ: D	constraint $\boldsymbol{\chi}=0$: -2	total 3D

3. How the gauge transformation properties of the new variables are determined to achieve the expected one?

   [If $\mathbf{n}(x)$ transforms in the adjoint representation under the gauge transformation, $f_{\mu\nu}(x)$ becomes gauge invariant.]

All of these problems have been simultaneously solved as follows.

• The reduction of enlarged gauge symmetry $G \times G/\tilde{H}$ to the original one G:
[K.K.,Murakami & Shinohara, hep-th/0504107, Prog.Theor.Phys.115, 201-216 (2006)]
For a given Yang-Mills field $\mathbf{A}_\mu(x)$, the color field $\mathbf{n}(x)$ is obtained by minimizing

$$F_{\rm rc} = \int d^D x \frac{1}{2}(D_\mu[\mathbf{A}]\mathbf{n}(x)) \cdot (D_\mu[\mathbf{A}]\mathbf{n}(x))$$

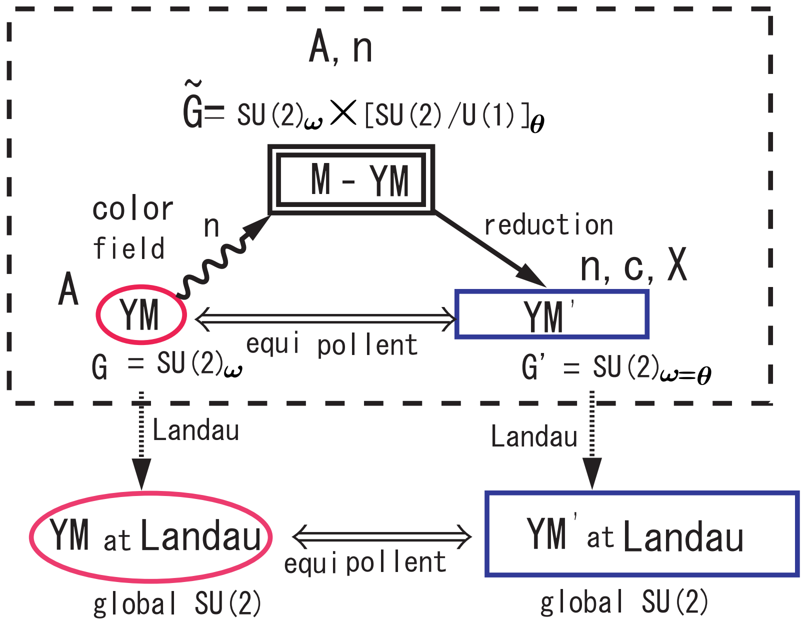
• The Jacobian associated with the change of variables:
[K.-I.K., Phys.Rev.D74, 125003 (2006)]

$$[d\mathscr{A}] = [dn][dc][dX]J$$

$J = 1$ by a suitable choice of the basis for X_μ^A

A new viewpoint of the Yang–Mills theory

$$\delta_\theta \mathbf{n}(x) = g\mathbf{n}(x) \times \theta(x) = g\mathbf{n}(x) \times \theta_\perp(x)$$



$$\delta_\omega \mathbb{A}_\mu(x) = D_\mu[\mathbb{A}]\omega(x)$$

By introducing a color field, the original Yang-Mills (YM) theory is enlarged to the master Yang–Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G}$. By imposing the reduction condition, it is reduced to the equipollent Yang-Mills theory (YM') with the gauge symmetry G'. The overall gauge fixing condition can be imposed without breaking color symmetry, e.g. Landau gauge.
[K.-I.K., Murakami & Shinohara, hep-th/0504107; Prog.Theor.Phys. **115**, 201 (2006).]
[K.-I.K., Murakami & Shinohara, hep-th/0504198; Eur.Phys.C**42**, 475 (2005)](BRST)

As a reduction condition, we propose minimizing the functional $\int d^D x \frac{1}{2} g^2 \mathbb{X}_\mu^2$ w.r.t. enlarged gauge transformations:

$$\min_{\omega,\theta} \int d^D x \frac{1}{2} g^2 \mathbb{X}_\mu^2 = \min_{\omega,\theta} \int d^D x (D_\mu[\mathbb{A}]\boldsymbol{n})^2. \tag{1}$$

Then the infinitesimal variation reads

$$0 = \delta_{\omega,\theta} \int d^D x \frac{1}{2} \mathbb{X}_\mu^2 = -\int d^D x (\boldsymbol{\omega}_\perp - \boldsymbol{\theta}_\perp) \cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu. \tag{2}$$

For $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$, the minimizing condition yields the differential form:

$$\boldsymbol{\chi} := D_\mu[\mathbb{V}]\mathbb{X}_\mu \equiv 0. \tag{3}$$

This denotes two conditions, since $\boldsymbol{n}(x) \cdot \boldsymbol{\chi}(x) = 0$ (following from $\boldsymbol{n}(x) \cdot \mathbb{X}_\mu(x) = 0$). For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$, the minimizing condition imposes no constraint.

Therefore, if we impose the reduction condition to the master-Yang–Mills theory, $\tilde{G} := SU(2)_\omega \times [SU(2)/U(1)]_\theta$ is broken down to the (diagonal) subgroup: $G' = SU(2)'$.

We have the equipollent Yang–Mills theory with the local gauge symmetry $G' := SU(2)^{\omega'}_{local}$ with $\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_{\parallel}(x), \boldsymbol{\omega}_{\perp}(x) = \boldsymbol{\theta}_{\perp}(x))$.

$$G = SU(2)^{\omega}_{local} \uparrow \tilde{G} := SU(2)^{\omega}_{local} \times [SU(2)/U(1)]^{\theta}_{local} \downarrow G' := SU(2)^{\omega'}_{local} \qquad (4)$$

The reduction condition has another expression in the differential form:

$$gD_{\mu}[\mathbb{V}]\mathbb{X}_{\mu} = gD_{\mu}[\mathbb{A}]\mathbb{X}_{\mu} = D_{\mu}[\mathbb{A}]\{\mathbf{n} \times (D_{\mu}[\mathbb{A}]\mathbf{n})\} = \mathbf{n} \times (D_{\mu}[\mathbb{A}]D_{\mu}[\mathbb{A}]\mathbf{n}) = 0 \qquad (5)$$

Thus, $\mathbf{n}(x)$ is determined by solving this equation for a given $\mathbb{A}_{\mu}(x)$. This determines the color field $\boldsymbol{n}(x)$ as a functional of a given configuration of $\mathbb{A}_{\mu}(x)$.

• Comparison between MAG and reduction condition:

Old MAG leaves local $U(1)_{local}(\subset G = SU(2)_{local})$ and global U(1)$_{global}$ unbroken, but breaks global SU(2)$_{global}$.

The reduction condition leaves local G'=SU(2)$_{local}$ and global SU(2)$_{global}$ unbroken (color rotation invariant)

The MAG in the original formulation is equivalent to set $n(x) \equiv (0, 0, 1)$ (a gauge fixing) in the new formulation.

⊙ Gauge transformation of new variables:

$$\delta_{\omega'}\mathbf{n} = g\mathbf{n} \times \boldsymbol{\omega'}, \tag{6a}$$

$$\delta_{\omega'}c_\mu = \mathbf{n} \cdot \partial_\mu \boldsymbol{\omega'}, \tag{6b}$$

$$\delta_{\omega'}\mathbb{X}_\mu = g\mathbb{X}_\mu \times \boldsymbol{\omega'}, \tag{6c}$$

$$\Longrightarrow \delta_{\omega'}\mathbb{V}_\mu = D_\mu[\mathbb{V}]\boldsymbol{\omega'} \Longrightarrow \delta_{\omega'}\mathbb{A}_\mu = D_\mu[\mathbb{A}]\boldsymbol{\omega'}, \tag{6d}$$

$$\Longrightarrow \delta_{\omega'}\mathbb{F}_{\mu\nu}[\mathbb{V}] = g\mathbb{F}_{\mu\nu}[\mathbb{V}] \times \boldsymbol{\omega}', \tag{6e}$$

Hence, the inner product $f_{\mu\nu} = \mathbf{n} \cdot \mathbb{F}_{\mu\nu}[\mathbb{V}]$ is SU(2)′ invariant.

$$\delta_{\omega'} f_{\mu\nu} = 0, \quad f_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}), \quad c_\mu = \mathbf{n} \cdot \mathbb{A}_\mu. \tag{7}$$

and $f_{\mu\nu}^2 = \mathbb{F}_{\mu\nu}[\mathbb{V}]^2$ is SU(2)′ invariant: SU(2) invariant "Abelian" gauge theory!

$$\delta_{\omega'}\mathbb{F}_{\mu\nu}[\mathbb{V}]^2 = \delta_{\omega'} f_{\mu\nu}^2 = 0. \tag{8}$$

Therefore, we can define the *gauge-invariant* monopole current by $k^\mu(x) := \partial_\nu {}^* f^{\mu\nu}(x) = (1/2)\epsilon^{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma}(x)$, Moreover,

$$\delta_{\omega'}\mathbb{X}_\mu^2 = 0. \tag{9}$$

Chapter:
Wilson loop
and
magnetic monopole

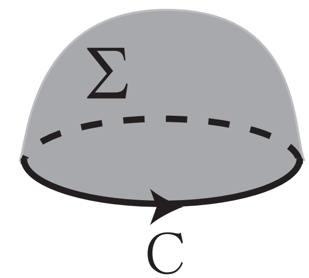
§ Wilson loop and magnetic monopole

⊙ Non-Abelian Stokes theorem for the Wilson loop

The Wilson loop operator for SU(2) Yang-Mills connection

$$W_C[\mathscr{A}] := \mathrm{tr}\left[\mathscr{P}\exp\left\{ig\oint_C dx^\mu \mathscr{A}_\mu(x)\right\}\right]/\mathrm{tr}(\mathbf{1}), \quad \mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)\sigma^A/2$$

The path-ordering $\mathscr{P}$ is removed by a non-Abelian Stokes theorem for the Wilson loop operator in the J representation of SU(2): $J = 1/2, 1, 3/2, 2, \cdots$
[Diakonov & Petrov, PLB 224, 131 (1989); hep-th/9606104]



$$W_C[\mathscr{A}] := \int d\mu_S(\boldsymbol{n}) \exp\left\{iJg\int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu} f_{\mu\nu}\right\}, \text{ no path-ordering}$$

$$f_{\mu\nu}(x) := \partial_\mu[\mathscr{A}_\nu^A(x)\boldsymbol{n}^A(x)] - \partial_\nu[\mathscr{A}_\mu^A(x)\boldsymbol{n}^A(x)] - g^{-1}\epsilon^{ABC}\boldsymbol{n}^A(x)\partial_\mu\boldsymbol{n}^B(x)\partial_\nu\boldsymbol{n}^C(x),$$

$$n^A(x)\sigma^A := U^\dagger(x)\sigma^3 U(x), \quad U(x) \in SU(2) \quad (A, B, C \in \{1, 2, 3\})$$

and $d\mu_S(\boldsymbol{n})$ is the product measure of an invariant measure on SU(2)/U(1) over S:

$$d\mu_S(\boldsymbol{n}) := \prod_{x\in S} d\mu(\boldsymbol{n}(x)), \quad d\mu(\boldsymbol{n}(x)) = \frac{2J+1}{4\pi}\delta(\boldsymbol{n}^A(x)\boldsymbol{n}^A(x) - 1)d^3\boldsymbol{n}(x).$$

$\odot$ The geometric and topological meaning of the Wilson loop operator [K.-I.K., arXiv:0801.1274, Phys.Rev.D77:085029 (2008)] [K.-I.K., hep-th/0009152]

$$W_C[\mathscr{A}] = \int d\mu_\Sigma(U) \exp\left\{ iJg(\Xi_\Sigma, k) + iJg(N_\Sigma, j) \right\}, \quad C = \partial\Sigma$$

$$k := \delta^* f = {}^*df, \quad \Xi_\Sigma := \delta^* \Theta_\Sigma \triangle^{-1} \leftarrow \quad \text{(D-3)-forms}$$

$$j := \delta f, \quad N_\Sigma := \delta \Theta_\Sigma \triangle^{-1} \leftarrow \quad \text{1-forms (D-indep.)}$$

$$\Theta_\Sigma^{\mu\nu}(x) = \int_\Sigma d^2 S^{\mu\nu}(x(\sigma)) \delta^D(x - x(\sigma))$$

k and j are gauge invariant and conserved currents, $\delta k = 0 = \delta j$.

The magnetic monopole is a topological object of co-dimension 3.
D=3: 0-dimensional point defect $\to$ point-like agnetic monopole (cf. Wu-Yang type)
D=4: 1-dimensional line defect $\to$ magnetic monopole loop (closed loop)

We do not need to use the Abelian projection ['t Hooft,1981] to define magnetic monopoles in Yang-Mills theory!

The Wilson loop operator knows the (gauge-invariant) magnetic monopole!

For $D=3$,

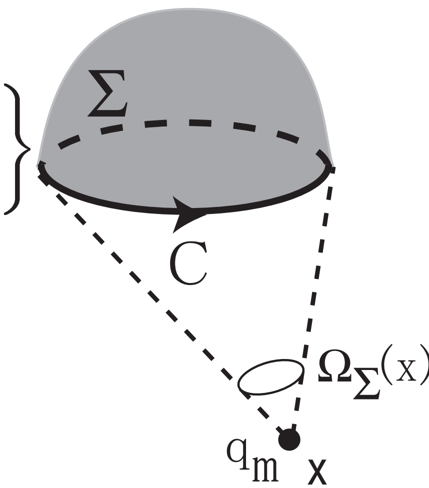
$$k(x)=\frac{1}{2}\epsilon^{jk\ell}\partial_\ell f_{jk}(x)=\rho_m(x)$$

denotes the magnetic charge density at x, and

$$\Xi_\Sigma(x)=\Omega_\Sigma(x)/(4\pi)$$

agrees with the (normalized) solid angle at the point x subtended by the surface Σ bounding the Wilson loop C. The magnetic part reads

$$W_{\mathscr{A}}^m := \exp\{iJg(\Xi_\Sigma,k)\} = \exp\left\{iJg\int d^3x\rho_m(x)\frac{\Omega_\Sigma(x)}{4\pi}\right\}$$



The magnetic charge q_m obeys the Dirac-like quantization condition :

$$q_m := \int d^3x\rho_m(x) = 4\pi g^{-1}n \quad (n\in\mathbb{Z})$$

[Proof] The non-Abelian Stokes theorem does not depend on the surface Σ chosen for spanning the surface bounded by the loop C,
See [K.-I.K., arXiv0801.1274, Phys.Rev.D77:085029 (2008)]

For $D=4$, the magnetic part reads using $\Omega_\Sigma^\mu(x)$ is the 4-dim. solid angle

$$W_{\mathscr{A}}^m = \exp\left\{ iJg \int d^4x \Omega_\Sigma^\mu(x) k^\mu(x) \right\}$$

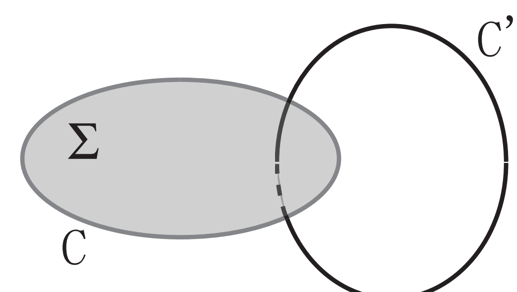
Suppose the existence of the ensemble of magnetic monopole loops C'_a,

$$k^\mu(x) = \sum_{a=1}^n q_m^a \oint_{C'_a} dy_a^\mu \delta^{(4)}(x - x_a), \quad q_m^a = 4\pi g^{-1} n_a$$

$$\Longrightarrow W_{\mathscr{A}}^m = \exp\left\{ iJg \sum_{a=1}^n q_m^a L(\Sigma, C'_a) \right\} = \exp\left\{ 4\pi J i \sum_{a=1}^n n_a L(\Sigma, C'_a) \right\}, \quad n_a \in \mathbb{Z}$$

where $L(\Sigma, C')$ is the linking number between the surface Σ and the curve C'.

$$L(\Sigma, C') := \oint_{C'} dy^\mu(\tau) \Xi_\Sigma^\mu(y(\tau))$$



where the curve C' is identified with the trajectory of a magnetic monopole and the surface Σ with the world sheet of a hadron (meson) string for a quark-antiquark pair.

Chapter:
Lattice reformulation of Yang-Mills theory and numerical simulations

§ Lattice formulation and numerical simulations

• Non-compact lattice formulation
[Kato, K.K., Murakami, Shibata, Shinohara and Ito, hep-lat/0509069, Phys.Lett.B 632, 326-332 (2006).]

· generation of color field configuration → Figure

· restoration of color symmetry (global gauge symmetry) → Figure

· gauge-invariant definition of magnetic monopole charge

• Compact lattice formulation:
[Ito, Kato, K.K., Murakami, Shibata and Shinohara, hep-lat/0604016, Phys.Lett.B 645, 67-74 (2007).]

· magnetic charge quantization subject to Dirac condition $gg_m/(4\pi) \in \mathbb{Z}$ → Table

· magnetic monopole dominance in the string tension → Table

[Shibata, Kato, K.K., Murakami, Shinohara and Ito, arXiv:0706.2529[hep-lat], Phys. Lett. B653, 101 (2007).]

$M_X = 1.2 \sim 1.3\text{GeV}$ ($M_A = 0.6\text{GeV}$? in the Landau gauge) → Figure

• color field configuration

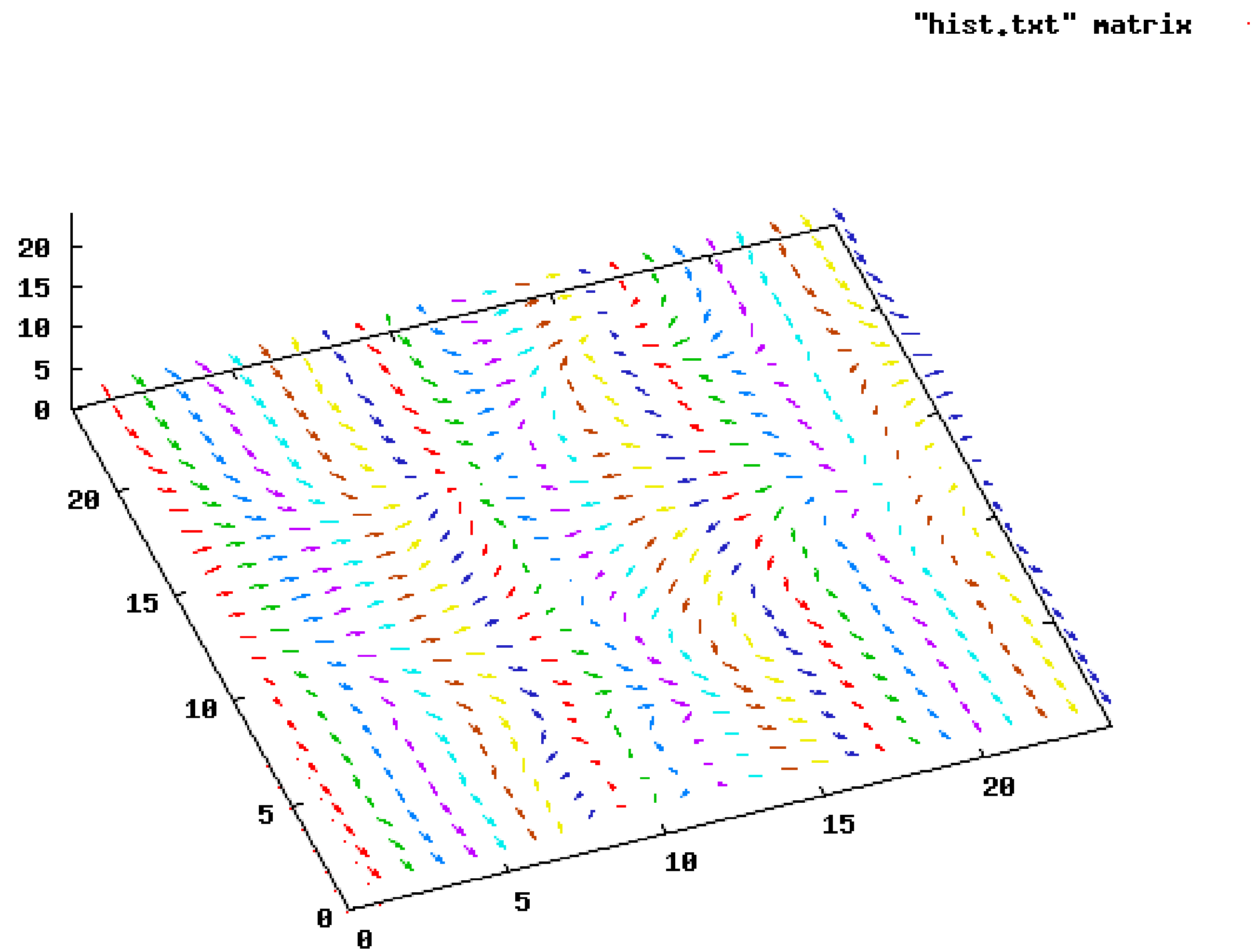


Figure 1: hedgehog (?) configurations of color field in SU(2) Yang-Mills theory

• Color symmetry (restoration) and the dynamical color field

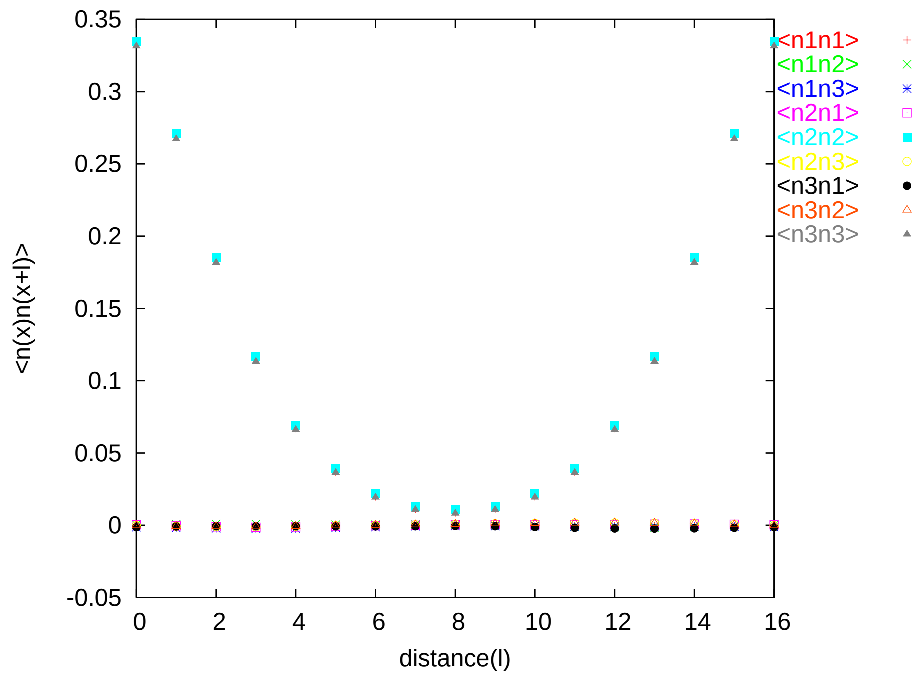


Figure 2: The plots of two-point correlation functions $\langle n_x^A n_0^B \rangle$ for $A, B = 1, 2, 3$ along the lattice axis on the 16^4 lattice at $\beta = 2.4$.
[Kato, K.K., Murakami, Shibata, Shinohara and Ito, hep-lat/0509069]

$$\langle n_x^A \rangle = 0 \quad (A = 1, 2, 3).$$
$$\langle n_x^A n_0^B \rangle = \delta^{AB} D(x) \quad (A, B = 1, 2, 3).$$

The global $SU(2)$ symmetry (color symmetry) is unbroken in our simulations.

- Magnetic charge quantization:

$$K(s,\mu) := 2\pi k_\mu(s) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\nu\bar{\Theta}_{\rho\sigma}(x+\mu),$$

Table 1: Histogram of the magnetic charge (value of $K(s,\mu)$) distribution for new and old monopoles on 8^4 lattice at $\beta = 2.35$.

Charge	Number(new monnopole)	Number(old monopole)
-7.5∼-6.5	0	0
-6.5∼-5.5	299	0
-5.5∼-4.5	0	1
-4.5∼-3.5	0	19
-3.5∼-2.5	0	52
-2.5∼-1.5	0	149
-1.5∼-0.5	0	1086
-0.5∼0.5	15786	13801
0.5∼1.5	0	1035
1.5∼2.5	0	173
2.5∼3.5	0	52
3.5∼4.5	0	16
4.5∼5.5	0	0
5.5∼6.5	299	0
6.5∼7.5	0	0

• String tension: magnetic monopole dominance

$$W_m(C) = \exp\left\{2\pi i \sum_{s,\mu} k_\mu(s)\Omega_\mu(s)\right\},$$

$$\Omega_\mu(s) = \sum_{s'} \Delta_L^{-1}(s-s')\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha S^J_{\beta\gamma}(s'+\hat{\mu}), \quad \partial'_\beta S^J_{\beta\gamma}(s) = J_\gamma(s), \qquad (1)$$

$$V_i(R) = -\log\{\langle W_i(R,T)\rangle / \langle W_i(R,T-1)\rangle\} = \sigma_i R - \alpha_i/R + c_i \quad (i=f,m), \quad (2)$$

Table 2: String tension and Coulomb coefficient I

β	σ_f	α_f	σ_m	α_m
2.4(8^4)	0.065(13)	0.267(33)	0.040(12)	0.030(34)
2.4(16^4)	0.075(9)	0.23(2)	**0.068(2)**	0.001(5)

Table 3: String tension and Coulomb coefficient II
MAG+DeGrand–Toussaint (reproduced from [Stack et al., PRD 50, 3399 (1994)]

β	σ_f	α_f	σ_{DTm}	α_{DTm}
2.4(16^4)	0.072(3)	0.28(2)	**0.068(2)**	0.01(1)

• quark-antiquark potential

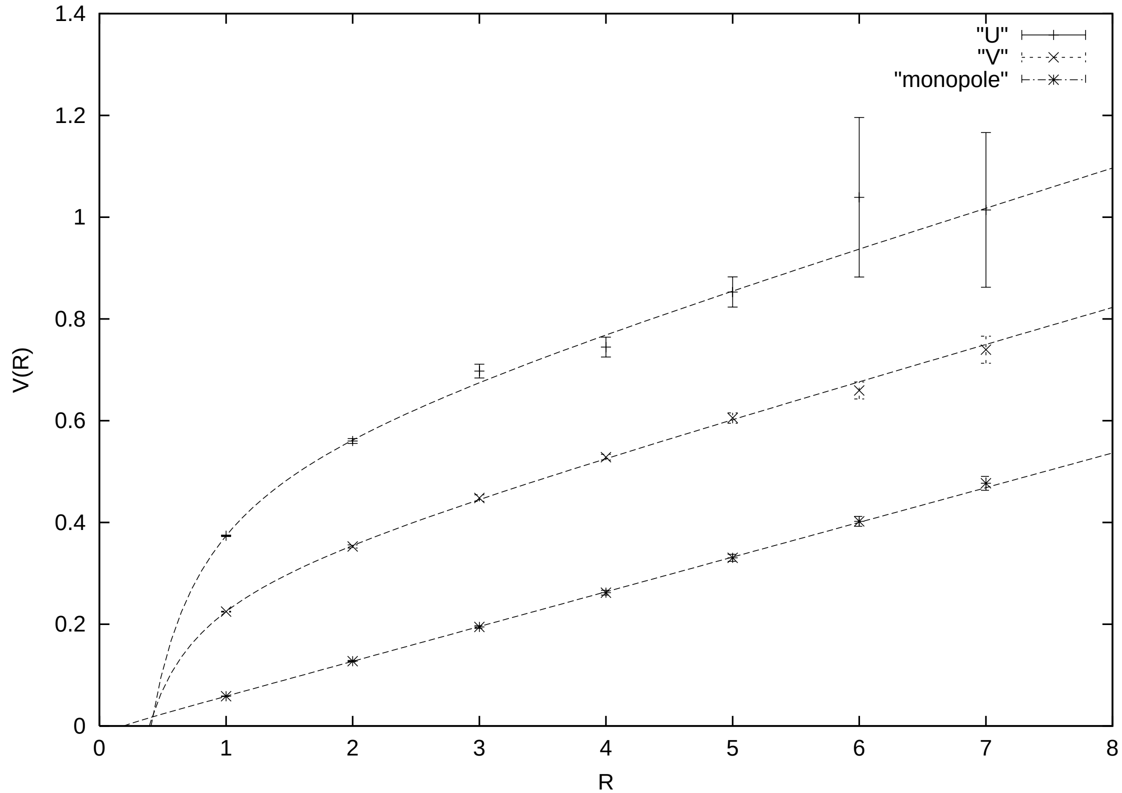


Figure 3: The full SU(2) potential $V_f(R)$, "Abelian" potential $V_a(R)$ and the magnetic–monopole potential $V_m(R)$ as functions of R at $\beta = 2.4$ on 16^4 lattice.
monopole part[Ito, Kato, K.K., Murakami, Shibata and Shinohara, hep-lat/0604016]
"Abelian" part[in preparation]

Table 4: String tension and Coulomb coefficient

β	σ_f	α_f	σ_{DTm}	α_{DTm}	σ_a	α_a
2.4(16^4)	**0.072(3)**	0.28(2)	**0.068(2)**	0.01(1)	**0.071(3)**	0.12(1)

- magnetic-monopole loops

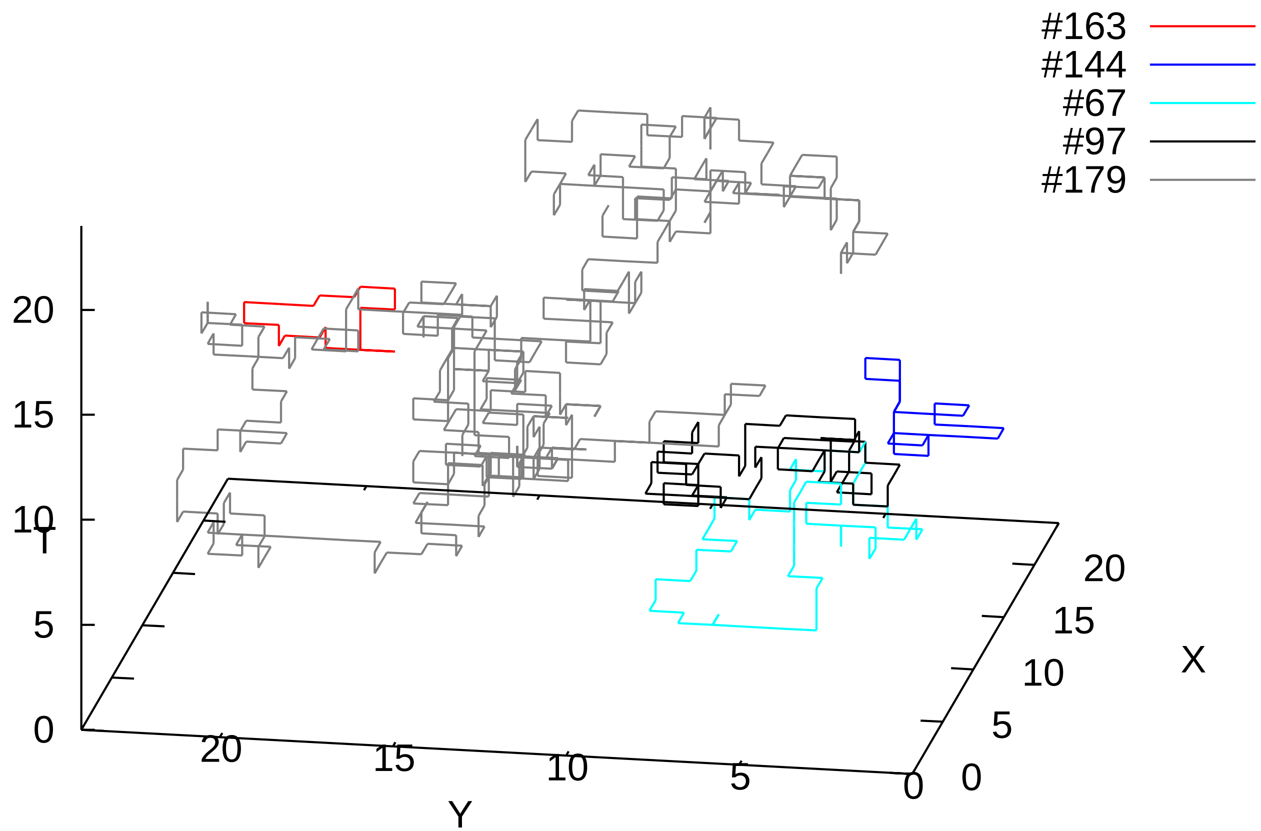


Figure 4: The magnetic-monopole loops on the 4 dimensional lattice where the 3-dimensinal plot is obtained by projecting the 4-dimensional dual lattice space to the 3-dimensional one, i.e., $(x, y, z, t) \to (x, y, t)$.

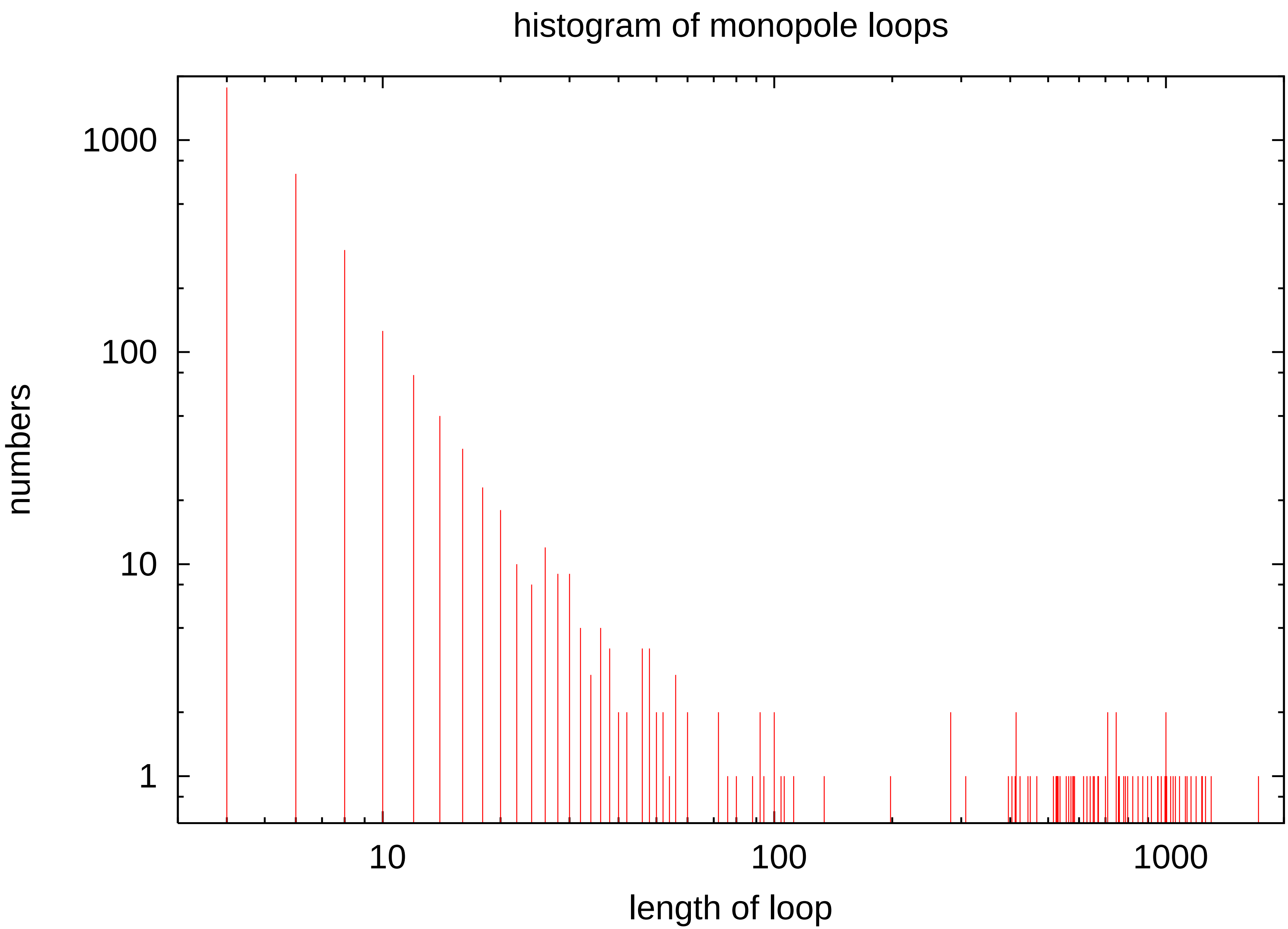


Figure 5: The number vs. length of the magneic monopole loops

- Two-point gluon correlation functions

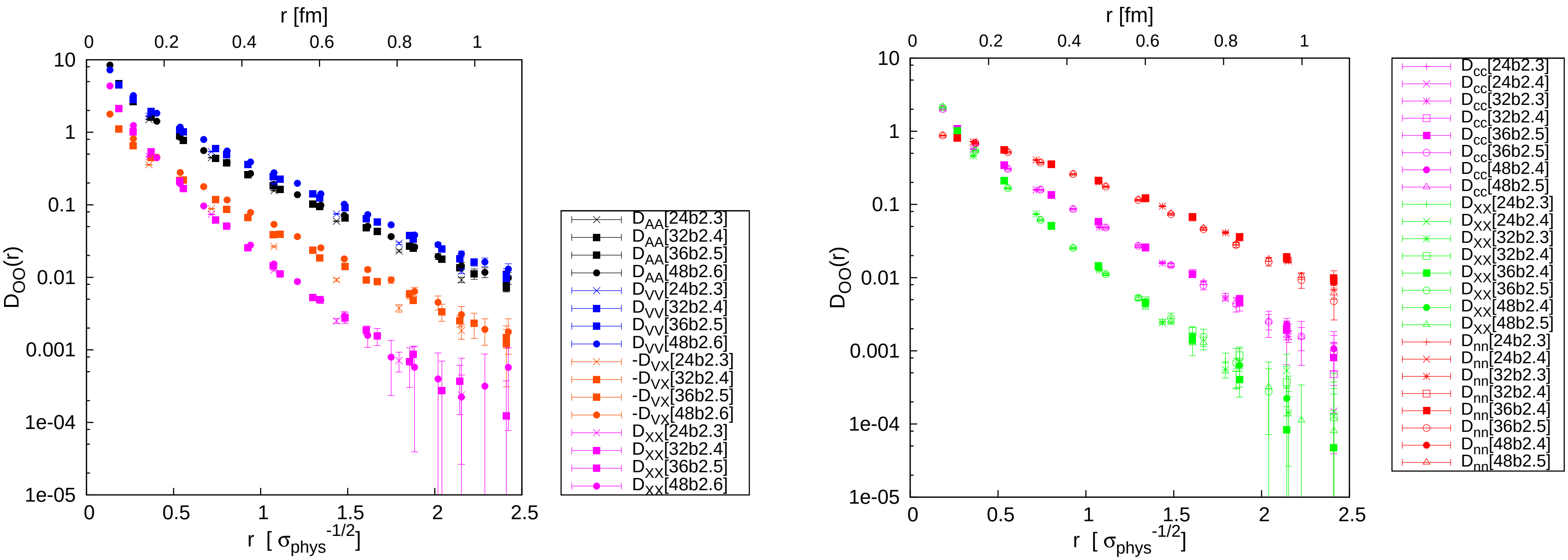


Figure 6: Logarithmic plots of scalar-type two-point correlation functions $D_{OO'}(r) := \langle \mathcal{O}(x)\mathcal{O}'(y)\rangle$ as a function of the Euclidean distance $r := \sqrt{(x-y)^2}$ for $\mathcal{O}$ and $\mathcal{O}'$. (Left panel) $\mathcal{O}(x)\mathcal{O}'(y) = \mathbb{V}_\mu^A(x)\mathbb{V}_\mu^A(y)$, $\mathbb{A}_\mu^A(x)\mathbb{A}_\mu^A(y)$, $-\mathbb{V}_\mu^A(x)\mathbb{X}_\mu^A(y)$, $\mathbb{X}_\mu^A(x)\mathbb{X}_\mu^A(y)$, (Right panel) $\mathcal{O}(x)\mathcal{O}'(y) = \mathbf{n}^A(x)\mathbf{n}^A(y)$, $c_\mu(x)c_\mu(y)$, $\mathbb{X}_\mu^A(x)\mathbb{X}_\mu^A(y)$, from above to below using data on the 24^4 lattice ($\beta = 2.3, 2.4$), 32^4 lattice ($\beta = 2.3, 2.4$), 36^4 lattice ($\beta = 2.4, 2.5$), and 48^4 lattice ($\beta = 2.4, 2.5, 2.6$). Here plots are given in the physical unit [fm] or in unit of square root of the string tension $\sqrt{\sigma_{\rm phys}}$.

[Shibata, Kato, K.K., Murakami, Shinohara and Ito, arXiv:0706.2529 [hep-lat]]

cf.[Amemiya and Suganuma, hep-lat/9811035] in mAG

• Infrared Abelian dominance

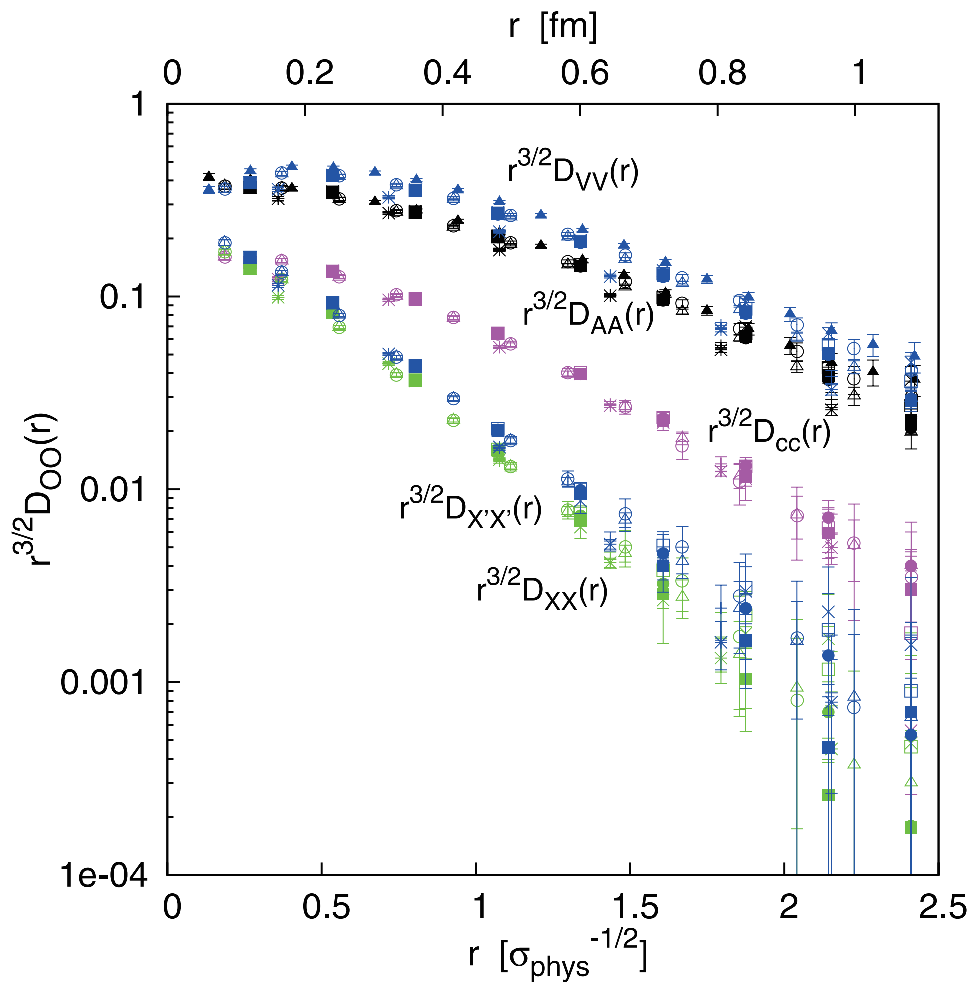
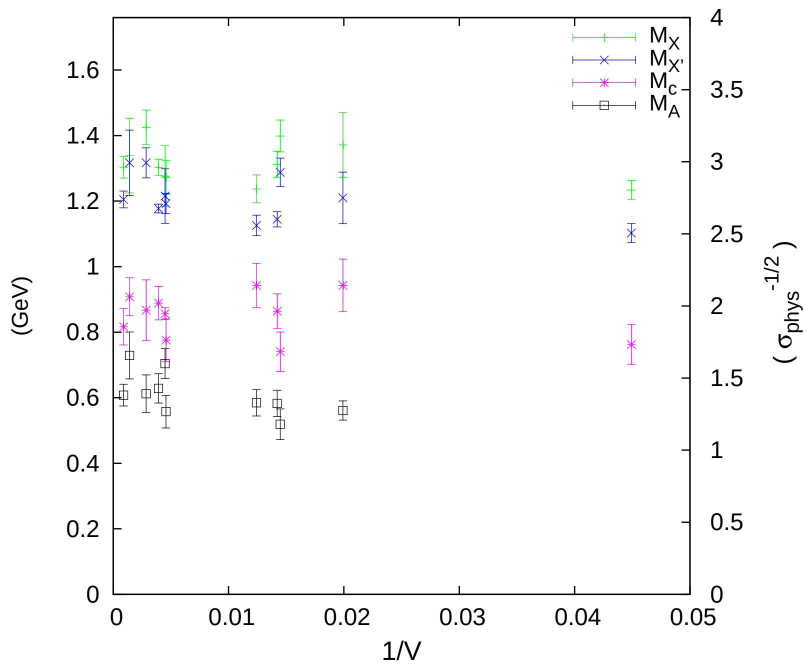


Figure 7: Logarithmic plots of the rescaled correlation function $r^{3/2}D_{OO}(r)$ as a function of r for $O = \mathbb{V}_\mu^A, \mathbb{A}_\mu^A, c_\mu, \mathbb{X}_\mu^A$ (and $\mathbb{X'}_\mu^A$) from above to below, using the same colors and symbols as those in Fig. 6. Here two sets of data for the correlation function $D_{XX}(x-y)$ are plotted according to the two definitions of the $\mathbb{X}_\mu^A$ field on a lattice.

• Gluon "mass" generation



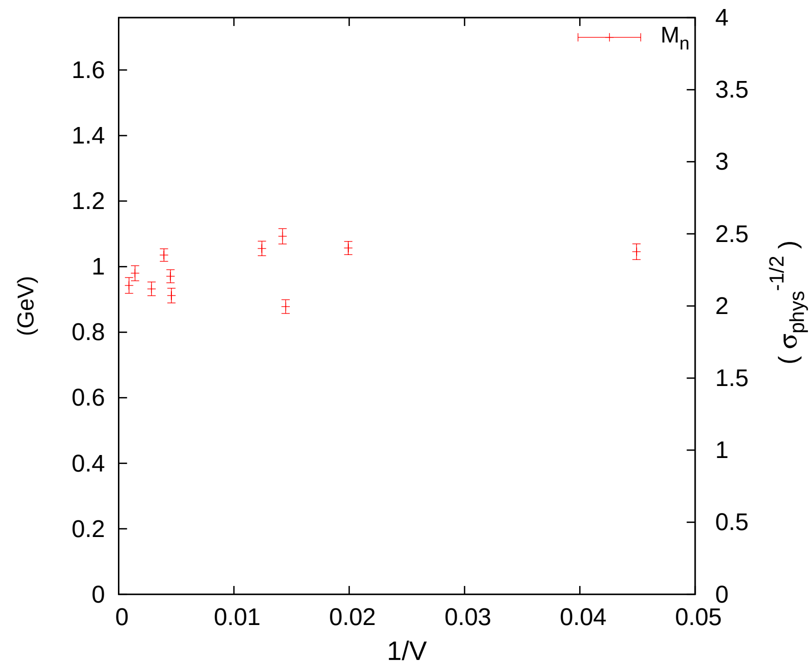


Figure 8: Gluon "mass" and decay rates (in units of GeV and $\sqrt{\sigma_{\rm phys}}$) as the function of the inverse lattice volume $1/V$ in the physical unit. (Left panel) for $\mathcal{O} = \mathbb{X}_\mu^A, (\mathbb{X}'{}_\mu^A), c_\mu, \mathbb{A}_\mu^A$ from above to below extracted according to the fitting: $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim r^{-3/2}\exp(-M_{\mathcal{O}}r)$, (Right panel) for $\mathbf{n}^A(x)$ extracted according to the fitting: $\langle \mathbf{n}^A(x)\mathbf{n}^A(y)\rangle \sim \exp(-M_n r)$.

$$M_X \simeq 2.98\sqrt{\sigma_{\rm phys}} \simeq 1.31\text{GeV},$$
$$M_{X'} \simeq 2.69\sqrt{\sigma_{\rm phys}} \simeq 1.19\text{GeV}. \tag{3}$$

$$M_n \simeq 2.24\sqrt{\sigma_{\rm phys}} \simeq 0.986\text{GeV},$$
$$M_c \simeq 1.94\sqrt{\sigma_{\rm phys}} \simeq 0.856\text{GeV},$$
$$M_A \simeq 1.35\sqrt{\sigma_{\rm phys}} \simeq 0.596\text{GeV}. \tag{4}$$

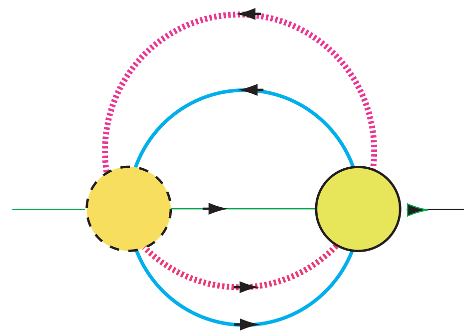
	lattice spacing ϵ		lattice size L [fm]			
β	$[1/\sqrt{\sigma_{phys}}]$	[fm]	24^4	32^4	36^4	48^4
2.3	0.35887	0.1609	3.8626	5.1501	5.7939	7.7252
2.4	0.26784	0.1201	2.8828	3.8438	4.3242	5.7657
2.5	0.18551	0.08320	1.9967	2.6622	2.9950	3.9934
2.6	0.13455	0.06034	1.4482	1.9309	2.1723	2.8964

Table 5: The lattice spacing ϵ and the lattice size L of the lattice volume L^4 at various value of β in the physical unit [fm] and the unit given by $\sqrt{\sigma_{phys}}$.

Chapter:
The relationship between magnetic monoole and instanton, merons, ...

§ Magnetic loops exist in the topological sector of YM_4

In the four-dimensional Euclidean SU(2) Yang-Mills theory, we have given a first* analytical solution representing circular magnetic monopole loops joining two merons:
[K.-I. K., Fukui, Shibata & Shinohara, arXiv:0806.3913, Phys.Rev.D78,065033 (2008)]



Our method reproduces also the previous results based on MAG (MCG) and LAG:
(i) A magnetic straight line can be obtained in the one-instanton or one-meorn background. $\to$ It disappears in the infinite volume limit.
[Chernodub & Gubarev, hep-th/9506026, JETP Lett. **62**, 100 (1995).]
[Reinhardt & Tok, hep-th/0011068, Phys.Lett.B**505**, 131 (2001). hep-th/0009205.]

(ii) A magnetic closed loop can NOT be obtained in the one-instanton background.
[Brower, Orginos & Tan, hep-th/9610101, Phys.Rev.D **55**, 6313–6326 (1997)]
[Bruckmann, Heinzl, Vekua & Wipf, hep-th/0007119,Nucl.Phys.B**593**, 545–561 (2001)]

*[Bruckmann & Hansen, hep-th/0305012, Ann.Phys.**308**, 201–210 (2003)] $Q_P = \infty$

§ What are merons?

	instanton	meron
discovered by	BPST 1975	DFF 1976
$D_\nu \mathscr{F}_{\mu\nu} = 0$	YES	YES
self-duality $*\mathscr{F} = \mathscr{F}$	YES	NO
Topological charge Q_P	$(0), \pm 1, \pm 2, \cdots$	$(0), \pm 1/2, \pm 1, \cdots$
charge density D_P	$\frac{6\rho^4}{\pi^2}\frac{1}{(x^2+\rho^2)^4}$	$\frac{1}{2}\delta^4(x-a) + \frac{1}{2}\delta^4(x-b)$
solution $\mathscr{A}_\mu^A(x)$	$g^{-1}\eta^A_{\mu\nu}\frac{2(x-a)_\nu}{(x-a)^2+\rho^2}$	$g^{-1}\left[\eta^A_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2} + \eta^A_{\mu\nu}\frac{(x-b)_\nu}{(x-b)^2}\right]$
Euclidean	finite action $S_{\rm YM} = (8\pi^2/g^2)\lvert Q_P\rvert$	(logarithmic) divergent actio
tunneling	between $Q_P = 0$ and $Q_P = \pm 1$ vacua in the $\mathscr{A}_0 = 0$ gauge	$Q_P = 0$ and $Q_P = \pm 1/2$ vacua in the Coulomb gauge
multi-charge solutions	Witten, 't Hooft, Jackiw-Nohl-Rebbi, ADHM	??? not known
Minkowski	trivial	everywhere regular finite, non-vanishing action

An instanton dissociates into two merons?

§ Relevant works (excluding numerical simulations)

papers	original configuration	dual counterpart	method
CG95	one instanton	a straight magnetic line	MAG (analytical)
BOT96	one instanton	no magnetic loop	MAG (numerical)
BHVW00	one instanton	no magnetic loop	LAG (analytical)
RT00	one meron	a straight magnetic line	LAG (analytical)
BOT96	instaton-antiinstanton	a magnetic loop	MAG (numerical)
	instaton-instaton	a magnetic loop	MAG (numerical)
RT00	instaton-antiinstanton	two magnetic loops	LAG (numerical)
Ours KFSS08	one instanton	no magnetic loop	New (analytical)
0806.3913	one meron	a straight magnetic line	New (analytical)
[hep-th]	two merons	circular magnetic loops	New (analytical)

CG95=Chernodub & Gubarev, [hep-th/9506026], JETP Lett. **62**, 100 (1995).
BOT96=Brower, Orginos & Tan, [hep-th/9610101], Phys.Rev.D **55**, 6313–6326 (1997).
BHVW00=Bruckmann, Heinzl, Vekua & Wipf, [hep-th/0007119], Nucl.Phys.B **593**, 545–561 (2001). Bruckmann, [hep-th/0011249], JHEP 08, 030 (2001).
RT00=Reinhardt & Tok, Phys.Lett. B**505**, 131–140 (2001). hep-th/0009205.
BH03=Bruckmann & Hansen, [hep-th/0305012], Ann.Phys. **308**, 201–210 (2003).

We solved the reduction: For a given Yang-Mills field $\mathbf{A}_\mu(x)$, minimize

$$F_{\rm rc} = \int d^D x \frac{1}{2} (D_\mu[\mathbf{A}]\mathbf{n}(x)) \cdot (D_\mu[\mathbf{A}]\mathbf{n}(x))$$

The local minimum is obtained by solving the reduction differential equation (RDE):

$$\mathbf{n}(x) \times D_\mu[\mathbf{A}] D_\mu[\mathbf{A}]\mathbf{n}(x) = \mathbf{0}.$$

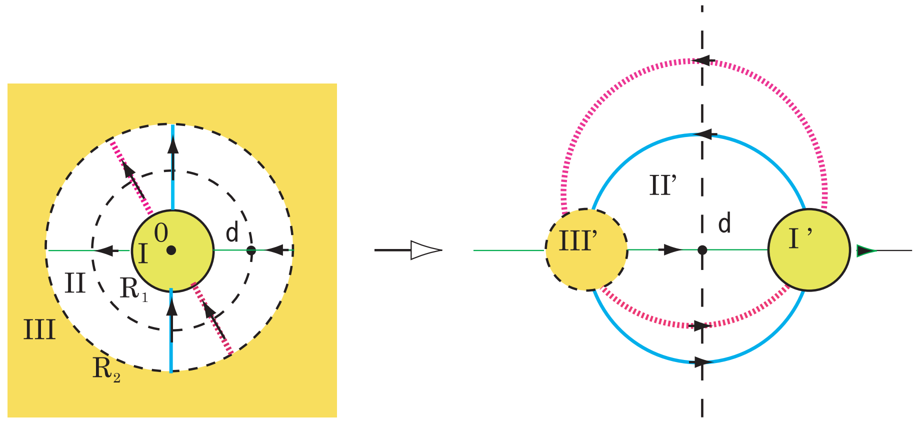
We consider a pair of merons at $x = a$ and $x = -a$

$$\mathbf{A}_\mu^{\rm MM}(x) = g^{-1} \left[\eta^A_{\mu\nu} \frac{(x+a)_\nu}{(x+a)^2} + \eta^A_{\mu\nu} \frac{(x-a)_\nu}{(x-a)^2} \right] \frac{\sigma_A}{2},$$

topological charge density

$$D_P(x) := \frac{1}{16\pi^2} \mathrm{tr}(\mathbf{F}_{\mu\nu} * \mathbf{F}_{\mu\nu}) = \frac{1}{2}\delta^4(x+a) + \frac{1}{2}\delta^4(x-a)$$

smeared meron pair of Callan, Dashen, Gross $\rightarrow$ conformal transformation + singular gauge transformation



The analytical solution representing a loop of magentic monopole: Using the conformal transformation and the singular gauge transformation,

$$\bar{\mathbf{n}}(x)_{\mathrm{II}'} = \frac{2a^2}{(x+a)^2}\hat{b}_\nu \eta^A_{\mu\nu} z_\mu U^{-1}(x+a)\sigma_A U(x+a)/\sqrt{z^2-(\hat{b}\cdot z)^2},$$

where

$$z_\mu = 2a^2\frac{(x+a)_\mu}{(x+a)^2} - a_\mu, \quad U(x+a) = \frac{\bar{e}_\alpha (x+a)_\alpha}{\sqrt{(x+a)^2}},$$

One-instanton limit: $|R_1 - R_2| \downarrow 0$ $(R_2/R_1 \downarrow 1)$. $S_{\mathrm{YM}}^{\mathrm{sMM}} = \frac{8\pi^2}{g^2}$ finite

One-meron limit: $R_2 \uparrow \infty$ or $R_1 \downarrow 0$ $(R_2/R_1 \uparrow \infty)$. $S_{\mathrm{YM}}^{\mathrm{sMM}}$ logarithmic divergence

Chapter:
Some open question
SU(2) case
(Preliminary results)

§ Numerical search for magnetic monople loops (Preliminary)

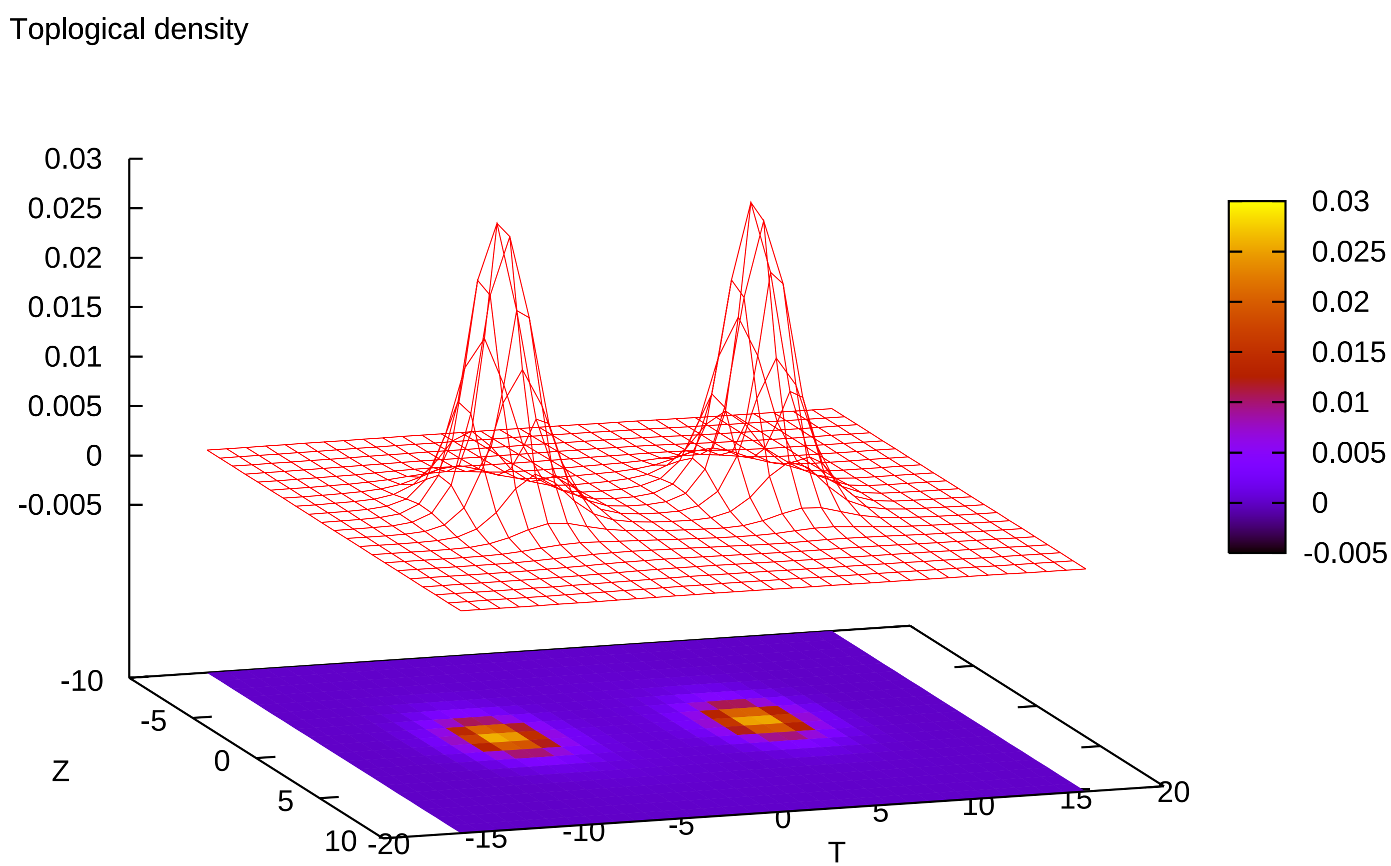


Figure 9: The plot of marginalized topological index density P(z,t) generated by a pair of (smeared) merons in 4-dimensional Euclidean space, where plot is obtained by the projection to z-t plane by integrated out for x and y variables (marginal-distribution).

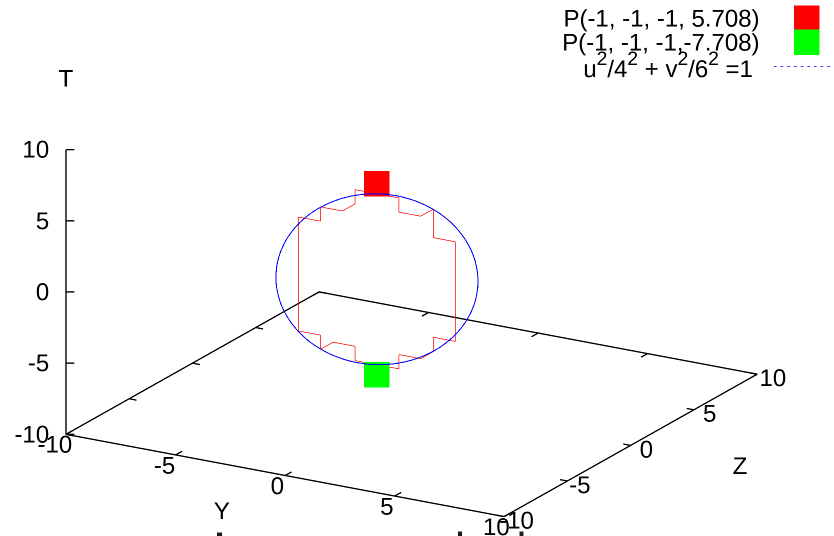


Figure 10: The plot of a magnetic-monopole loop generated by a pair of (smeared) merons in 4-dimensional Euclidean space where the 3-dimensinal plot is obtained by projecting the 4-dimensional dual lattice space to the 3-dimensional one, i.e., $(x, y, z, t) \to (y, z, t)$. The positions of two meron sources are described by solid boxes, and the monopole loop by red solid line. In the lattice of the volume [-10, 10]$^3\times$ [-16, 16] with a lattice spacing $\epsilon = 1$, the two-merons are located at $(-1, -1, -1, -1 \pm 6.078)$, and are smeared with the instanton cap of size $R = 3.0$ ($d = 12$, $R1 = 2.833$ and $R2 = 50.833$). The monopole loop is confined in the 3-dim. space $x = -1$ and in a 2-dim. plane rotated about t-axis by 0.46rad. (For guiding the eye, the monopole loop is fitted by an ellipsoid curve (blue dotted line) with the long radius 6 and the short radius 4.)

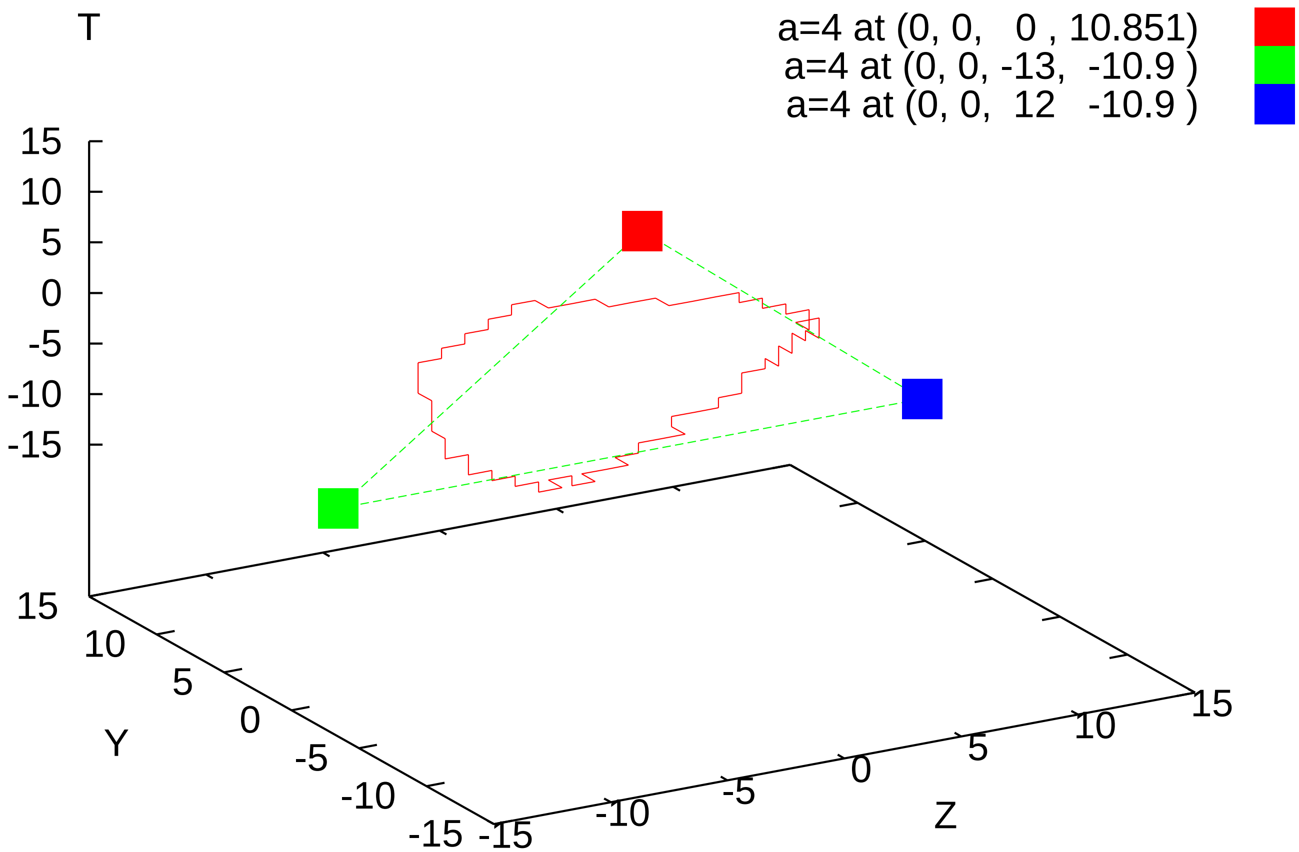


Figure 11: The 3-dimensional projection of a magnetic-monopole loop generated by two-instanton of the JNR type in 4 dimensional lattice $[-15,15]^2 \times [-30,30]^2$ with its spacing $\epsilon = 1$. The monopole loop is written by a red solid curve and the two-instanton solution is parametrized by the "size" a and the "position" denoted by a box: $a = 4$ at $(0,0,0,10.851)$, $a = 4$ at $(0,0,-13.-10.9)$, $a = 4$ at $(0,0,12,-10.9)$.

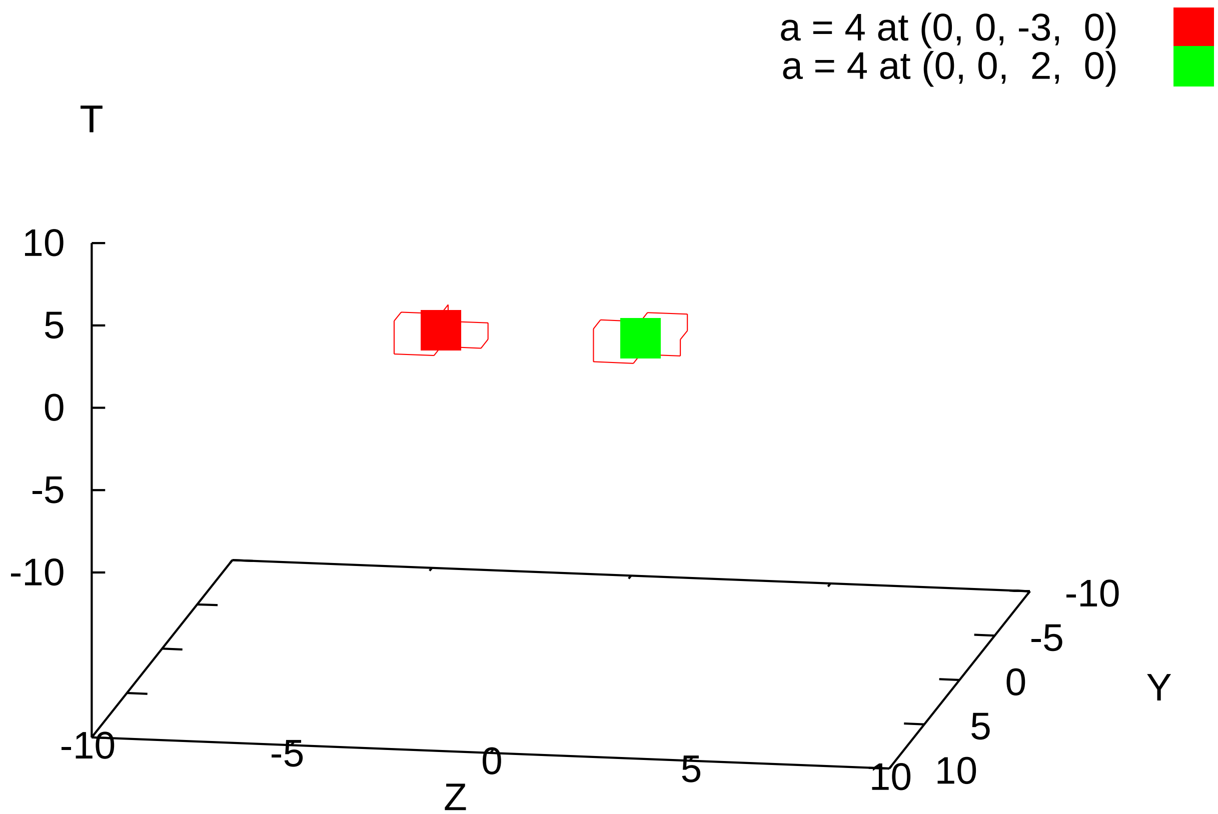


Figure 12: The 3-dimensional projection of two collapsed magnetic-monopole loops generated by the two-instanton of the 't Hooft type in 4 dimensional lattice $[-12,12]^2 \times [-20,20]^2$ with its spacing $\epsilon = 1$. The monopole loop is written by a red solid curve and the two-instanton solution is parametrized by the "size" a and the "position" denoted by a box: $a = 4$ at $(0,0,-3,0)$ $a = 4$ at $(0,0,2,0)$.

§ Adjoint quark potential and String breaking

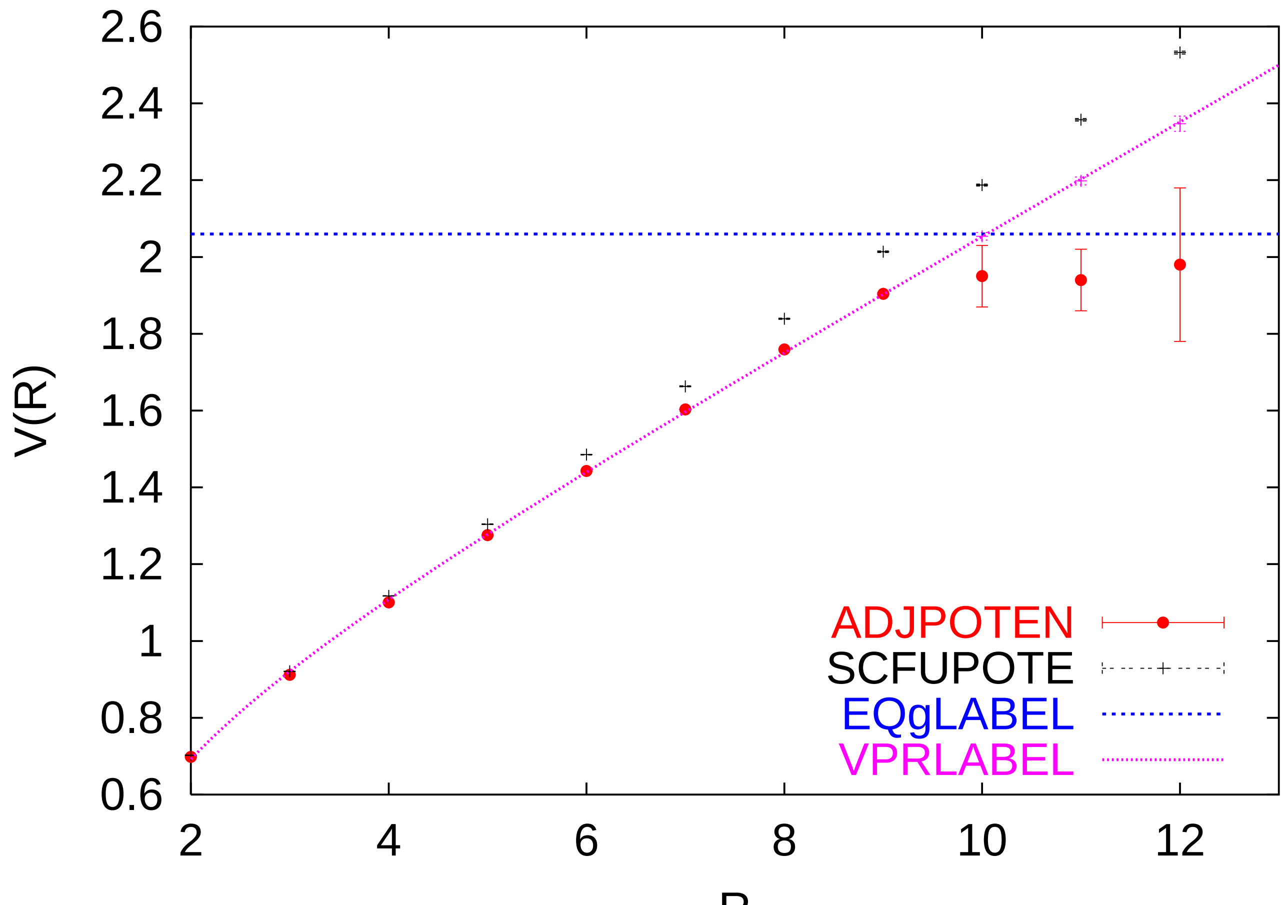


Figure 13: S. Kratochvila and Ph. de Forcrand, String breaking with Wilson loops?, hep-lat/0209094, Nucl.Phys.Proc.Suppl.119:670-672,2003
D=3, G=SU(2); The adjoint and $\frac{8}{3}$ fundamental static potentials $V(R)$ vs R. The horizontal line at 2.06(1) represents twice the energy of a gluelump.

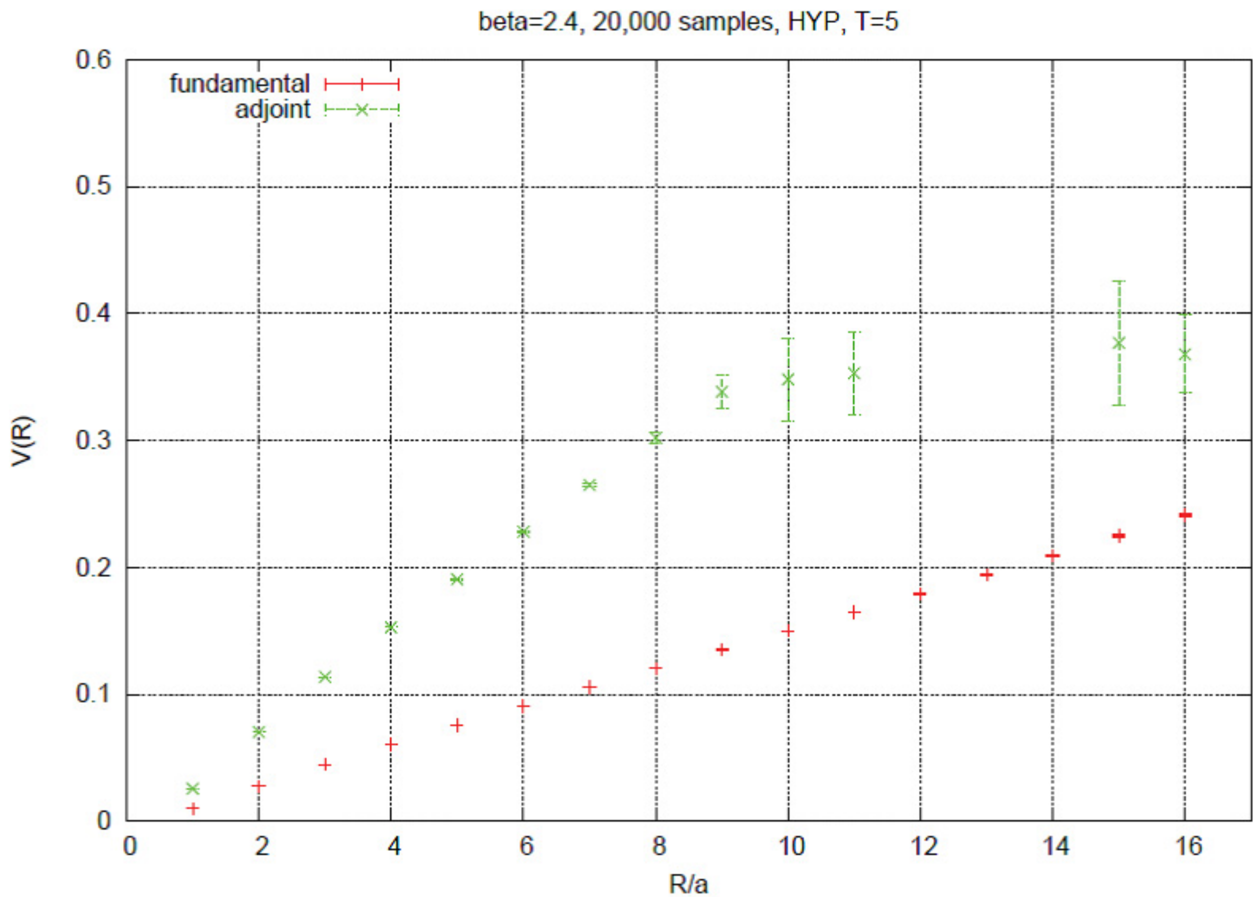


Figure 14: Our preliminary result.

- Abelian dominance in the adjoint Wilson loop? $\Rightarrow$Casimir scaling, string breaking
- monopole dominance in the adjoint Wilson loop?

For the ensemble of point-like magnetic charges:

$$k(x) = \sum_{a=1}^{n} q_m^a \delta^{(3)}(x - z_a)$$

$$\Longrightarrow W_{\mathscr{A}}^m = \exp\left\{ iJ\frac{g}{4\pi}\sum_{a=1}^{n} q_m^a \Omega_\Sigma(z_a) \right\} = \exp\left\{ iJ\sum_{a=1}^{n} n_a \Omega_\Sigma(z_a) \right\}, \quad n_a \in \mathbb{Z}$$

The magnetic monopoles in the neighborhood of the Wilson surface Σ ($\Omega_\Sigma(z_a) = \pm 2\pi$) contribute to the Wilson loop

$$W_{\mathscr{A}}^m = \prod_{a=1}^{n} \exp(\pm i 2\pi J n_a) = \begin{cases} \prod_{a=1}^{n} (-1)^{n_a} & (J = 1/2, 3/2, \cdots) \\ = 1 & (J = 1, 2, \cdots) \end{cases}$$

$\Longrightarrow$ N-ality dependence of the asymptotic string tension
[K.-I. K., arXiv:0802.3829, J.Phys.G35:085001,2008]

§ Conclusion and discussion

The second method a la Cho & Faddeev-Niemi has been fully developped in the last decade:

• Path integral formulation is completed (action and measure for new field variables).

• The relevant lattice gauge formulation are available for numerical simulations.

In particular,

• The gauge-invariance of the magnetic monopole is guaranteed from the begining by construction.

• The direct relevance of the magnetic monopole to the Wilson loop and "Abelian" dominance in the operator level are manifest via a non-Abelian Stokes theore.

The second method have already reproduced all esssential results obtained so far by the first method, i.e., Abelian projection by 't Hooft.

• "Abelian " dominance in the string tension (Wilson loop average)

• magnetic monopole dominance in the string tension (Wilson loop average)

The first method (Abelian projection) is included as a special limit of the second mehtod (Cho & Faddeev-Niemi). The first method is nothing but a gauge-fixed version of the second method.

• Extending our results to SU(3):

· Continuum formulation
[K.-I. K., arXiv:0801.1274, Phys.Rev.D **77**, 085029 (2008)]
[K.-I. K., Shinohara & Murakami, arXiv:0803.0176, Prog.Theor.Phys.**120**, 1–50 (2008)]
For SU(3), there are two options for introducing the color field.
For the Wilson loop in the fundamental rep.,

$$\boldsymbol{n} \in G/\tilde{H} = SU(3)/U(2) \neq SU(3)/[U(1) \times U(1)]$$

Quarks in the fundamental rep. can be confined by a non-Abelian magnetic monopole described by a single color field for any N in $SU(N)$ against the Abelian projection scenario.

· Lattice formulation [K-I.K., Shibata, Shinohara, Murakami, Kato and Ito, arXiv:0803.2451 [hep-lat], Phys.Lett.B669, 107-118 (2008)]
Preliminary numerical simulations e-Print: arXiv:0810.0956 [hep-lat] (Lattice 2008)
non-Abelian magnetic monopole dominance in the string tension

• color confinement

It is desirable to make clear the relationship between color confinement in general and quark confinement based on dual superconductor picture. Our approach opens a path to investigate this issue, since we have recovered color symmetry in this approach of deriving the dual superconductor picture.

Thank you for your attention!

- SU(3) numerical simulations:

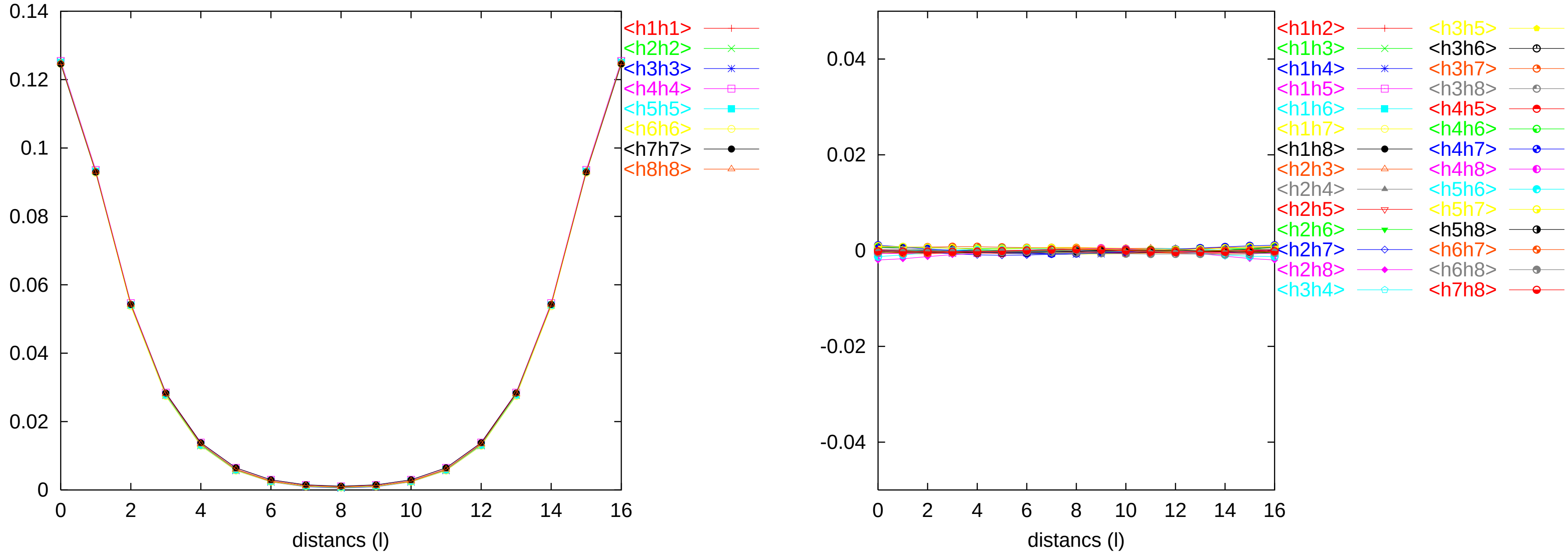


Figure 15: The correlation functions of the color field on the 16^4 lattice at $\beta = 5.7$. (Left panel) diagonal parts, (Right panel) off-diagonal parts.

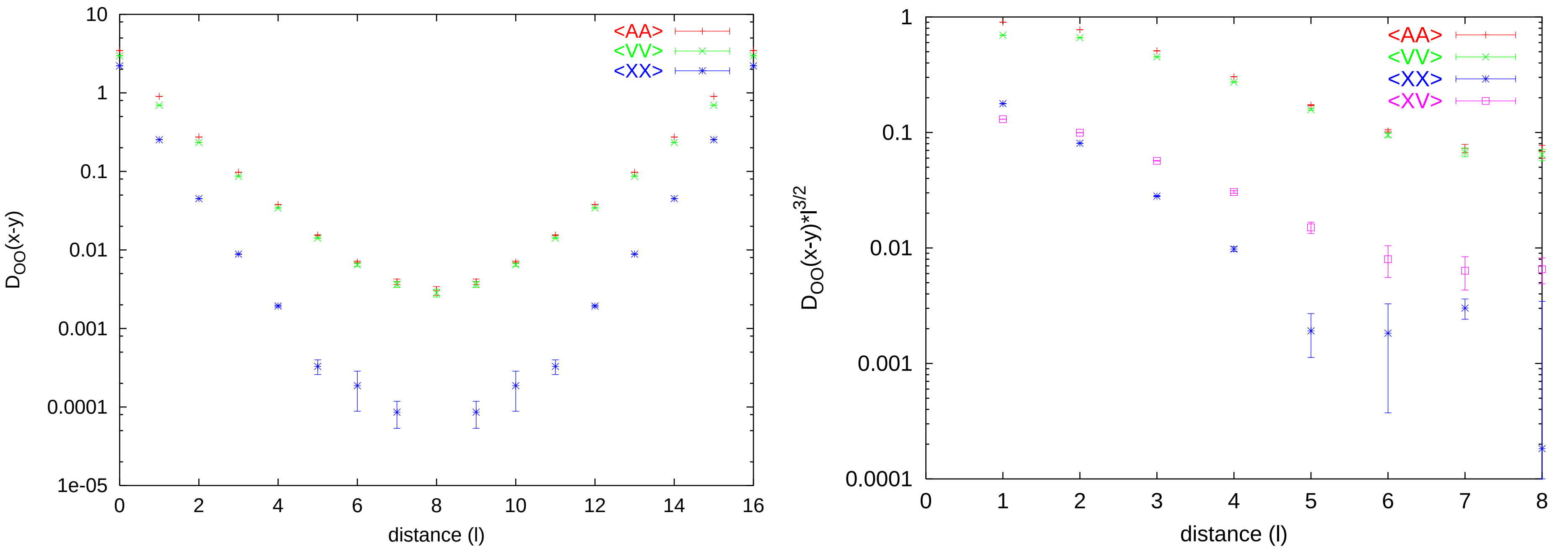


Figure 16: The correlation functions of the original gauge field and new variables on the 16^4 lattice at $\beta = 5.7$. (Left panel) The log. plot of correlation functions. (Right panel) The log. plot of the scaled correlation functions.

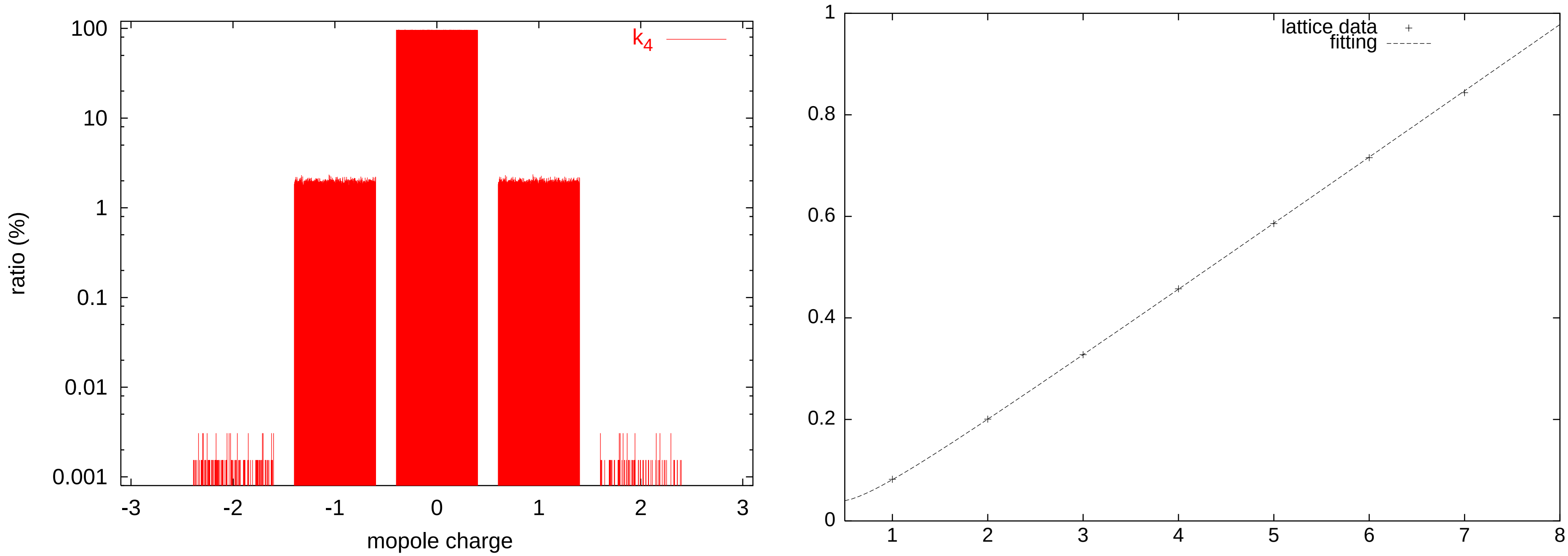


Figure 17: (Left panel) the magnetic monopole charge distribution, (Right panel) the static interquark potential calculated only from the magnetic monopole part on the 16^4 lattice at $\beta = 5.7$.

The numerical data of the static potential $V_m(R)$ is well fitted by a function $V_m(R) = -\alpha_m/R + \sigma_m R + \beta_m$ with the value $\sigma_m = 0.1137(12)$. In comparison with the full string tension $\sigma_{full} = (0.3879(39))^2 = 0.1505(30)$, thus, we have shown the non-Abelian magnetic monopole dominance for the string tension in the $SU(3)$ Yang-Mills theory:

$$\sigma_m/\sigma_{full} = 0.76. \tag{1}$$

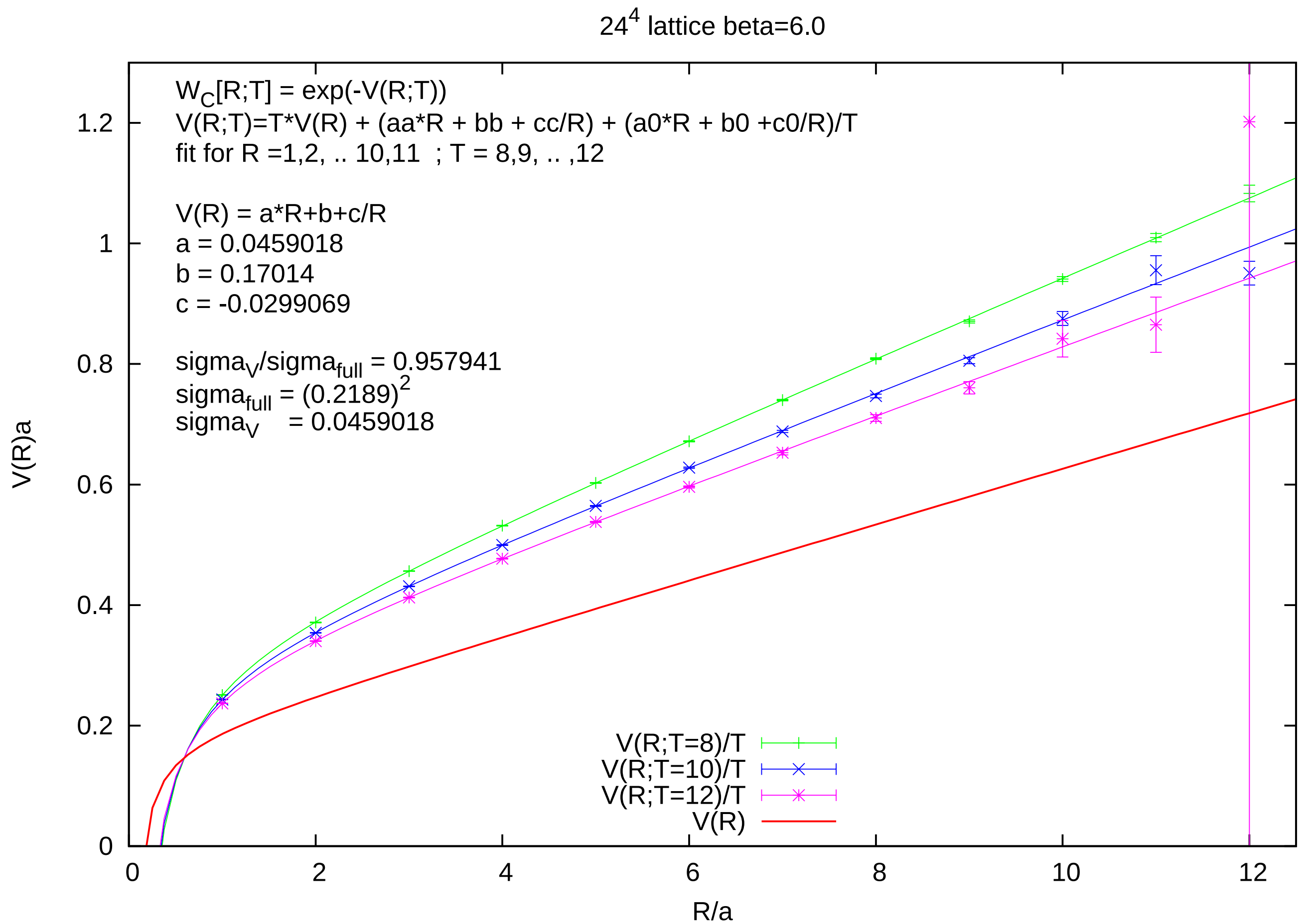


Figure 18: The static interquark potential calculated only from the V part on the 24^4 lattice at $\beta = 6.00$ using 500 samples. The V part leads to $\sigma_V/\epsilon(\beta)^2 = (0.214247/\epsilon(\beta))^2 = 0.0459018/\epsilon(\beta)^2$. The physical string tension is given by $\sigma_{full}/\epsilon(\beta)^2 = (0.2189/\epsilon(\beta))^2$. This shows the V(Abelian) dominance: $\sigma_V/\sigma_{full} = 0.957941$. The date for the full potential is extracted from R.G.Edward, U.M. Heller and T.R.Klassen, Nucl. Phys. B617, 377-392 (1998)[hep-lat/9711003v2].hep-lat/0103029.

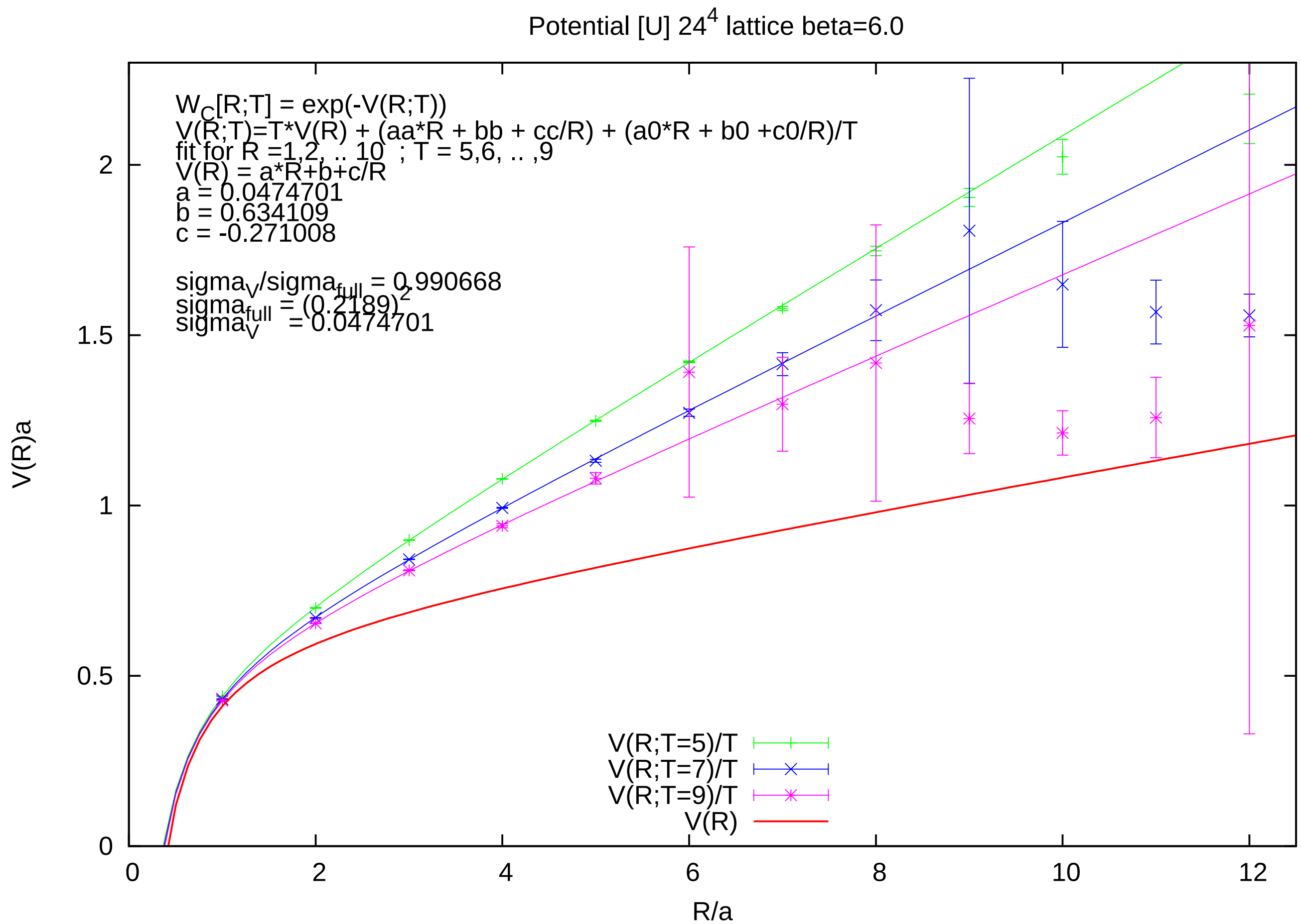


Figure 19: The static interquark potential calculated on the 24^4 lattice at $\beta = 6.00$ using 500 samples.

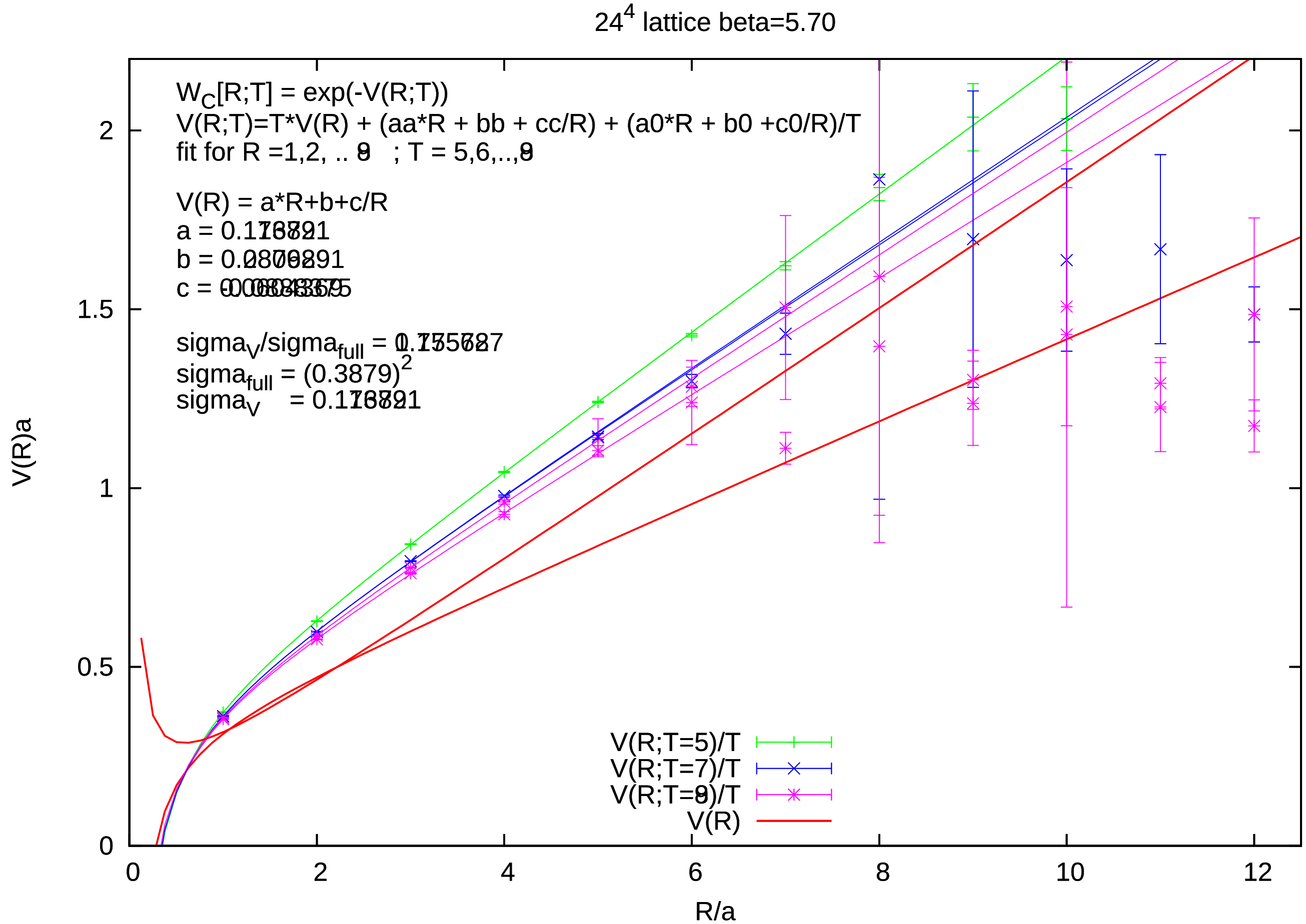


Figure 20: The static interquark potential calculated only from the V part on the 24^4 lattice at $\beta = 5.70$ using 500 samples. The V part leads to $\sigma_V/\epsilon(\beta)^2 = (0.214247/\epsilon(\beta))^2 = 0.0459018/\epsilon(\beta)^2$. The physical string tension is given by $\sigma_{full}/\epsilon(\beta)^2 = (0.2189/\epsilon(\beta))^2$. This shows the V(Abelian) dominance: $\sigma_V/\sigma_{full} = 0.957941$. The date for the full potential is extracted from R.G.Edward, U.M. Heller and T.R.Klassen, Nucl. Phys. B617, 377-392 (1998)[hep-lat/9711003v2].hep-lat/0103029.